DISPERSION ENGINEERING OF BOSE-EINSTEIN CONDENSATES

By

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To the Faculty of Washington State University:

The members of the Committee appointed to examine the dissertation of MOHAMMAD AMIN KHAMEHCHI find it satisfactory and recommend that it be accepted.

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done for me. I would like to thank my wife Yasmin for bringing happiness to my life, her constant support, patience, and kindness.
The subject of this dissertation is engineering the dispersion relation for dilute Bose-Einstein condensates (BECs). When a BEC is immersed into suitably tailored laser fields its dispersion can be strongly modified. Prominent examples for such laser fields include optical lattice geometries and Raman dressing fields. The ability to engineer the dispersion of a BEC allows for the investigation of a range of phenomena related to quantum hydrodynamics and condensed matter.

In the first context, this dissertation studies the excitation spectrum of a spin-orbit coupled (SOC) BEC. The spin-orbit coupling is generated by “dressing” the atoms with two Raman laser fields. The excitation spectrum has a Roton-like feature that can be altered by tuning the Raman laser parameters. It is demonstrated that the Roton mode can be softened, but it does not reach the ground state energy for the experimental conditions we had. Furthermore, the expansion of SOC BECs in 1D is studied by relaxing the trap allowing the BEC to expand in the SOC direction. Contrary to the findings for optical lattices, it is observed that the condensate partially occupies quasimomentum states with negative effective mass, and therefore an abrupt deceleration is observed although the mean field force is along the direction of expansion.

In condensed-matter systems, a periodic lattice structure often plays an important role. In this context, an alternative to the Raman dressing scheme can be realized by coupling the s- and p- bands of a static optical lattice via a weak moving lattice. The bands can be treated as pseudo-spin states. It is shown that similar to the dispersion relation of a Raman dressed SOC, the quasimomentum of the ground state is different from zero. Coherent coupling of the SOC dispersion minima can lead to the realization of the stripe phase even though it is not the thermodynamic ground state of
the system. Along the lines of studying the hydrodynamics of BECs, three novel multicomponent solitonic states are realized. It is shown that the solitons are structurally stable and the oscillation of vector dark-anti-dark solitons is studied in a weak harmonic trap.
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The energy spectrum of the single particle Hamiltonian shown in Eq. (3.13). The color of the curve represents the spin composition of the states. Spin composition is defined as $(|\psi_\uparrow|^2 - |\psi_\downarrow|^2) / (|\psi_\uparrow|^2 + |\psi_\downarrow|^2)$ in which $\psi_\uparrow$ and $\psi_\downarrow$ are the wave functions of the $|1, -1\rangle$ and $|1, 0\rangle$ atomic states, respectively.

(a) BdG spectrum (solid line) and single-particle dispersion (dotted line) of a spin-orbit coupled BEC for a nonlinear coefficient $g = 0.186$, Raman detuning $0.28E_R$ ($\delta = 2\pi \times 500$ Hz) and Raman coupling strength $2.5E_R$. (b) Mode softening with decreasing Raman detuning. The lines correspond to a Raman detuning $1E_R$, $0.5E_R$ and $0$ from top to bottom.

Roton-like minimum softening and energetic instability for decreasing Raman detuning $\delta$ in a miscible regime ($g_{11} = g_{22} = 0.186, g_{12} = 0.08$) with $\Omega = 2.5E_R$. The lines correspond to the Raman detuning of $0.6E_R$, $0.4E_R$, $0.2E_R$ and $0$ from top to bottom.
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(a) Bragg spectrum for a spin-orbit coupled BEC, measured for $\hbar \Omega = 3.5 E_R$ and $\delta = 2\pi \times 500$ Hz. Each point is an average over four measurements. (b) Schematic of the transitions corresponding to the three peaks in the spectrum. (c) Experimental ToF image of the condensate. The atoms diffracted to the left are due to the Bragg pulse. The number of diffracted atoms enclosed by the dashed ellipse is used for measuring the peak’s $\alpha$, $\beta$, $\gamma$ heights.

Mode softening. Position of Bragg peaks as a function of Raman detuning. Each point is an average over four data runs. This data was taken for $\hbar \Omega = 3.5 E_R$. Vertical error bars are on the order of the symbol size. The data quality in the uppermost branch is impacted by the smallness of the spin overlap between the initial and final state.

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(a) The experimental arrangement of the trap and the Raman beams is illustrated. The angle between the Raman beams is $90^\circ$. (b) The Raman coupling scheme in the $F = 1$ manifold is displayed. The $|1, +1\rangle$ state is out of resonance due to the quadratic Zeeman shift.

1D expansion of a BEC along the SOC direction ($x$ axis) is presented with the Raman coupling strength $\Omega = 2.5 \ E_{R}$, and the detuning $\delta$ is $1.36 \ E_{R}$. The positions of the edges of the BEC are shown as the BEC expands in the dipole beam. The purple (light orange) circles are experimental data indicating the positions of the right (left) cloud edges. The solid curves show the cloud edges found by the GP simulations. The shaded shapes schematically show the BEC expanding in the dipole trap as time advances and the dashed lines schematically follow the edges of the expanding cloud.

Symmetric expansion of a BEC along the axial direction ($x$ axis) with no Raman coupling and strong coupling ($4.5 \ E_{R}$) is presented. The solid lines are the solution to Eq. (3.18). Beyond $\Omega = 4 \ E_{R}$ the double well structure as well as the negative effective mass region disappear in the dispersion relation. The deviation from the theoretical prediction is due to the anharmonic nature of the dipole beam at larger $x$ coordinates.
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Comparison between the results of the 3D simulation and the experiments. The red dotted curves are the integrated densities, and the solid black curves are the integrated theoretical 3D results. The theoretical results are calculated for \( \delta = 2.73 \, E_R, \, 1.37 \, E_R, \) and \( 0.546 \, E_R \), respectively and the experimental results, from left to right, are measured for \( \delta = 2.71(3) \, E_R, \, 1.36(3) \, E_R, \) and \( 0.54(3) \, E_R \), respectively. There is a slight difference between the experimental and the theoretical detunings, however the theoretical values lie well within the experimental errorbars.

ToF image of a two component (|\( \uparrow \rangle \) and |\( \downarrow \rangle \)) BEC after being held in the dipole trap for 15 s. The species are separated using a vertical Stern-Gerlach field. This image is taken for one of our calibration measurements for the data presented in this section. The reason the contrast is lower than usual is that it is taken using the \( 1 \rightarrow 1' \) atomic transition.

Raman laser beams intensity simulation. (a) Numerical results of the normalized combined laser intensities of the Raman beams at the position of the atoms looked from the top (looking towards \(-z\) direction). (b) Calculated Raman laser intensities along the \( x \) axis. The combined Raman intensities fall below 90% for \(|x| > 39.2 \, \mu m\).

In-trap expansion of the condensate is shown on the left column for detunings \( 0.54 \, E_R \approx 1 \, kHz, \, 0.41 \, E_R \approx 750 \, Hz, \, 0.27 \, E_R \approx 500 \, Hz, \) and \( 0.14 \, E_R \approx 250 \, Hz \) from top to bottom, respectively. In the right column, a time slice of the 1D cloud density is presented.

Stern-Gerlach ToF absorption images of the condensate after (a) 8 ms and (b) 9 ms in-trap expansion.
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The quantiles with 5% increment are shown for an expanding BEC. The quantiles that are closer to the edges of the cloud contain more noise.

Comparison of the full two-component 1D GPE model Eq. (3.20) (top) with the 1D single-band model Eq. (3.22) (middle) and the axially symmetric 3D single-band model (bottom) for detuning $\delta = 0.546E_R$.

Effective single-particle dispersion Eq. (3.23) (solid lines, left axis) and spin–quasi-momentum mapping Eq. (3.21) (dotted lines, right axis) for the experimental parameters $\Omega = 2.5E_R (w = 0.63)$ and $\delta = \{2.73, 1.37, 0.546\}E_R (d = \{0.683, 0.341, 0.137\})$, respectively, from top to bottom. The two inflection points along with the middle point, which has the least negative effective mass are shown as dots with thin lines demonstrating the group velocities $v = \frac{\partial E_{\pm}(k)}{\partial k}$ shown in Fig. 57. The region of negative effective mass between the inflection points is lightly shaded.

Plot of the total density, $n = n_\uparrow + n_\downarrow$, as a function of time for the single-band 1D GPE simulation of the in-situ expansion described in the text. Top: Full image demonstrating the asymmetric expansion seen in the experiment. Bottom: Zoom into the region where the modulational instability first appears. The dashed lines are the three group velocities $v_g = E'_\pm(k)$ at the quasi-momenta $k$ where the effective mass $m_\pm^{-1} = E''_{\pm}(k)$ starts becoming negative (steepest line), the point of minimum negative effective mass (middle), and where the mass returns to positive (least steep line). The red dots indicate the region with negative effective mass.
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1D GPE simulations of in-situ expansion with different detunings $\delta = (2.73, 1.37, 0.546, 0.273) E_R [d = (0.683, 0.341, 0.137, 0.068)]$, respectively, from top to bottom. The detunings used in the simulations are within the experimental errorbars. The panels on the right are the zoomed-in version of the left panels. Red dots denote regions where the local quasi-momenta $k$ corresponds to negative effective mass $m^{-1}_s = E''_-(k) < 0$.

(a) Schematic of the experimental setup. (b) Raman coupling scheme in the $F = 1$ manifold. (c) SOC dispersions 1–4 are shown for $\Omega = 1, 1.5, 2, 2.7 E_R$, respectively, while $\delta = 500$ Hz. The arrows show the points of the dispersion which are coupled by the moving lattice. The color coding indicates the spin composition of the dressed states. (d) A typical experimental ToF image of the SOC condensate with the running lattice. This particular image is taken for $\Omega = 2.7 E_R$, $\delta = 250$ Hz, and $2 E_L$ moving lattice depth. Dashed rectangles $A$ and $C$ enclose the atoms that normally appear in the SOC ToF images while $B$ and $D$ represent those that have undergone the lattice two-photon transition.

Experimental results - Spin composition of the condensate vs. the moving lattice depth for (a) $\delta = 1000$ Hz, (b) $\delta = 500$ Hz, and (c) $\delta = 250$ Hz. Every circle represents a single measurement. (a) and (b) contain approximately 60 measurements and (c) contains approximately 90 measurements.
62 (a) Six states in the bare state basis and the coupling between them is schematically shown. (b) Occupation probability of the bare states as a function of time during the 50 ms lattice potential ramp. A realistic detuning $\delta$ is included in the simulation but it is not shown in the figure. (c) The equivalent states in the dressed states basis and the corresponding couplings between them are presented. (d) Occupation probability of the dressed states as a function of time during the lattice potential ramp.

63 (a) Spin composition $\langle \sigma_z \rangle$ as a function of time $t$ for SOC strength $\Omega = 2.7 E_R$ and lattice strength $V = 1 E_R$. (b) $\langle \sigma_z \rangle$ at $t = 50$ ms as a function of $\Omega$ at $V = 1 E_R$. (c) $\langle \sigma_z \rangle$ at $t = 50$ ms as a function of $V$ for $\Omega = 2.7 E_R$.

64 Adiabatic lattice ramp rate occurs at much slower rates compared to Raman coupling time scale. Here, the SOC coupling strength $\Omega$ is $2.7 E_R$ and the lattice is ramped up with the rate of $2 E_R / s$ for 500 ms.

65 Top: Ground-state spatial density (left) and the real-momentum (right) distributions of a SOC BEC in a typical trap and with interactions. The wavefunction is obtained from the numerical GP equations. Bottom: same distributions at $t = 50$ ms after the lattice potential linearly increases to $V = E_R$. The numbers denote corresponding states as in Fig. 62(a).

66 (a) Spin polarization $\langle \sigma_z \rangle$ vs time obtained from the GP simulations. (b) The population of the six largest momentum distribution peaks vs. time is presented. The labels ($P_1 - P_6$) shown in (b) correspond to peaks ($1 - 6$) shown in the bottom right panel of Fig. 65, respectively.
Atom loss curve close to the Feshbach resonance. Each point represents one measurement. The red curve is a Gaussian fit with center position of 9.14 G and width of 0.095 G. The nominal magnetic field is calculated based on a measured magnetic field close to 10 G using RF spectroscopy. Then the smaller magnetic fields are found by linear extrapolation based on the electric current of the little bias coils. The linearly extrapolated magnetic field might be slightly different from the actual magnetic field due to a slight nonlinearity in the current servo system.

Magnetic field dependence of the interspecies scattering length between the atomic states $|1, +1\rangle$ and $|2, -1\rangle$ is evaluated based on Eq. (3.33) based on the parameters estimated in [2]. (b) The critical Raman coupling strength $\Omega_C$ is calculated based on Eq. (3.34) for magnetic fields close to the Feshbach resonance with the scattering lengths evaluated using Eq. (3.33). This value is maximum at $B \approx 9.12$ G.

ToF images of the Ball-shell structure in a 50/50 mixture of $|1, +1\rangle$ and $|2, -1\rangle$ in the vicinity of the Feshbach resonance. The atomic clouds shown on the top and the bottom of each image are in the $|2, -1\rangle$ and $|1, +1\rangle$ atomic states, respectively.

Experimental setup and schematic lattice illustration. (a) Experimental arrangement. The crossed dipole trap beams propagate in the $e_x$ and $e_z$ directions. The static and moving lattices have overlapping beams propagating along $e_x + e_y$ and $-e_x + e_y$. The static lattice is generated using the red beams with laser frequency $\omega$, and moving lattice is generated using the blue beams with laser frequencies $\omega'$ and $\omega' + \delta\omega$. (b) Lattice potentials along the $e_x$ direction. The lattice period $d$ is identical for the static lattice $V_0$ and the moving lattice $V'_x$. The initial offset between lattice sites of the static and moving lattice, $\Delta x$, is dependent on the initial phase $\phi_0$ between the two lattices. (c) illustrates the lattice beam intensities as a function of space and time.
Experimental setup and schematic lattice illustration. (a,b) Illustration of the multi-photon processes for the driven lattice system and the corresponding FB band structure in the first Brillouin zone. The static lattice induces a large energy gap (I) through a 2-photon process and a small energy gap (II) through a 4-photon process. The moving lattice induces an energy gap when the $s$-band and the $p$-band are coupled through (III). A smaller energy gap is produced by a combination of the static and moving lattice (IV).

Effects of the driving frequency. (a) Band minimum $q_{\text{min}}$ for the upper hybrid band vs. driving frequency $\Delta \omega$. The depth of the moving lattice is $1 \ E_R$. The filled circles are experimental measurements with standard deviation errorbars. The black line shows the theoretical prediction of a two-band model. The squares and stars are the results of numerical simulations of the Schrödinger equation and the GP equation, respectively. (b) Calculated upper hybrid $s$-$p$ FB band structure for different driving frequencies $\Delta \omega = 4.99 \ E_R, 5.1 \ E_R$ and $5.22 \ E_R$ from top to bottom. The lowest (black) curve is the $s$ orbital band without the presence of the driving field.

Effects of the driving strength. Band minimum $q_{\text{min}}$ vs. driving field strength $V'$ for different driving frequencies of (a) $|\Delta \omega| = 2.92E_R$ and (c) $|\Delta \omega| = 5.21E_R$. The red points are experimental data with standard deviation errorbars, and the solid lines are the theoretical predictions from a two-band model. $\text{sgn}(\Delta \omega)$ determines the direction of motion of the moving lattice. (b,d) Corresponding hybrid band structures for different driving field strengths $V'_x = 1.5E_R, 0.75E_R$ and $0E_R$ (red, blue, and black curves respectively).
Plots of the evolution of the BEC momentum space density distribution as a function of $k_x$ for different evolution times. The moving lattice depth is ramped to (a) $V' = 0.25E_R$, (b) $V' = 0.5E_R$, (c) $V' = 1.0E_R$ in the first 60 ms and then keep this value for the following 40 ms. The white lines indicate the center of the small dipole oscillation from which the band minimum is determined. The driving frequency is $|\Delta \omega| = 5.21E_R$.

Observation of the first order phase transition in the condensate for moving lattice strengths larger than approximately $1.7 E_R$. (a) Shows the theoretical estimation of the dispersion relations starting from small moving lattice strengths to the larger strengths from the top to the bottom. (b) Shows the trace of the dispersion minima as a function of the moving lattice strength. The data points are the experimental results that are in excellent agreement with the theory.

Stability of the Floquet system. (a) Number of atoms remaining in trap after adiabatically loading a BEC into the FB band, normalized to initial atom number determined from independent experimental runs. The static lattice is ramped on to $5.47 E_R$ in 200 ms. Then the moving lattice is ramped on to a depth of $V' = 0.5 E_R$ in 60 ms. The dips $\alpha$, $\beta$, and $\gamma$ occur close to the Bloch bands $p$, $d$, and $f$. Panels (b), (c), and (d) show the TOF images and the effective band structures for data points labeled (b), (c), and (d) in panel (a). The arrows show where the BEC is situated after the moving lattice is adiabatically ramped on.

Non-calibrated BEC stability results showing the atom number versus the moving lattice velocity. (a) and (b) correspond the loss curves shown in Fig. 76 (a) left and right respectively.
Quench dynamics after suddenly jumping on the coupling between the $s$ and $p$ band. Band structure before (a) and after (b) jumping on the coupling between the $s$ and $p$ band. (c) Two examples for experimental images taken 0.5 ms after the quench. The difference between the images is due to the relative phase between the static and the moving lattice. The experimental data for panels (d) and (e) are extracted from such images by counting the atom number in the dashed boxes. They indicate the $-2\hbar k$ (left rectangle), $0$ (middle rectangle) and $+2\hbar k$ component (right rectangle). Panels (d) and (e) represent the normalized population of the momentum component $-2\hbar k$ and $0$ $\hbar k$ respectively. The spread of the experimental data is indicated by the height of the rectangles. The shaded areas are the results of numerical GP simulations calculated for a uniform spread of phases $\phi_0$. The black curves show the numerical result for $\phi_0 = 0$. Panel (f) presents the correlation between numerical and experimental spread for the data from (d) and (e). The solid line is a fit to the scatter.

Condensate lifetimes in the static lattice. (a) and (b) show the atom numbers in arbitrary units for $5E_R$ and $6.5E_R$ deep static lattices respectively.

Sample TOF images of a full Rabi oscillation between the $s$ and the $p$ bands is presented.

Dependence of the Bloch bands on the optical lattice depth. (a) Illustrates the Bloch bands for optical lattice depths 0-7 $E_R$. Black corresponds to 0 $E_R$. Red and Cyan correspond to $p$ and $s$ bands for 7 $E_R$ optical lattice depth. (b) The blue curve shows the band gap at the Brillouin zone edge as a function of the lattice height. The red line represents $y = \frac{1}{2}x$. 
Panel (a) presents the elongated cloud imaged in TOF with $|2, -2\rangle$ appearing on the top, and $|1, -1\rangle$ appearing on the bottom. The currents in the little bias coils one and two for this configuration are 8.965 A and 9.03 A, respectively. Panel (b) illustrates the image similar to (a) with little bias coil currents 8.855 A and 9.03 A for little bias one and two coils respectively.

(a) and (b) illustrate the atom loss curves as a function of hold time in the trap for $|1, -1\rangle$ and $|2, -2\rangle$ atomic states, respectively.

Experimental realization of dark-antidark solitary waves. (a) Absorption images (upper two panels) and corresponding integrated cross sections (lower two panels) of a dark-antidark solitary wave. The dark soliton component resides in a cloud of $|F, m_F\rangle = |2, -2\rangle$ atoms (upper and third panel), while the bright component consists of atoms in the $|F, m_F\rangle = |1, -1\rangle$ state (second and forth panel from top). (b) Experimentally observed oscillation of the dark-antidark solitary wave in the trap. The position is measured along the x-axis, i.e. along the weakly confining axis of the trap. The time is measured starting from the initial microwave pulse that creates the two-component mixture. The blue dots are experimental data, while the red line is a sine function fit to the data. (c) Comparison between a dark-bright soliton in a mixture of atoms in the $|F, m_F\rangle = |1, -1\rangle$ and $|F, m_F\rangle = |1, 0\rangle$ states (left image) and a dark-antidark structure in a mixture of atoms in the $|F, m_F\rangle = |2, -2\rangle$ and $|F, m_F\rangle = |1, -1\rangle$ states (right image).

DB solitons are formed using the $|1, -1\rangle$ and $|1, 0\rangle$ atomic states. The experimental procedure used for this figure is similar to that of dark-antidark solitons. For this particular image, there is 15 s wait time inside the trap.
1D Dark-Antidark solitary waves. Three examples of the two components in the
dark-antidark state for progressively increasing $g_{12}$: (a) in-trap case with trap fre-
quency $\Omega = 0.025 \ll 1$, (b) homogeneous case with $\Omega = 0$. (c) Dependence of the
lowest eigenfrequencies scaled by the trap frequency $\Omega$ for the in-trap case. The plot
of the real part of eigenvalues: black dots indicate the theoretical prediction for the
anomalous mode (see discussion in the text). The colors are there only to visually
aid the eye to identify the continuation of the different modes. Plot of the imaginary
part: evidence of very low growth of oscillatory instability for $0.71 < g_{12} < 0.87$.

Logarithmic scale of a Stern-Gerlach ToF image of a DBB soliton. There is a 100 ms
wait time after the DBB soliton is generated to show the stability of the structure.

Logarithmic scaled ToF image of DDB solitons after 100 ms wait time. The image
is taken using the Stern-Gerlach imaging procedure.

Versatility and robustness of DDB solitons is demonstrated. A large variation of the
relative atomic population is allowed between $|1, -1\rangle$ and $|1, 0\rangle$ atomic states. (a)
has 31%, 63%, and 6% population for $|1, -1\rangle$, $|1, 0\rangle$, and $|1, +1\rangle$ states, respectively.
(b) has 71%, 25%, and 4% population for $|1, -1\rangle$, $|1, 0\rangle$, and $|1, +1\rangle$ states, respectively.

The temperature controller schematic and the Printed Circuit Board are shown in
the figure. The schematic circuit design is by G. Forrest Cook, and I designed the
PCB using Eagle PCB design software.

Schematic of a simple atomic level scheme.

Schematic illustration of saturation spectroscopy.
I dedicate my dissertation to my mother and my father for years of their kind and enduring support.
CHAPTER 1

INTRODUCTION

In this dissertation, we use atomic dilute gas Bose-Einstein condensates (BECs) to simulate Hamiltonians that are relevant in solid-state physics and hydrodynamics. An atomic dilute gas BECs provide a reliable test bed for investigating a large variety of modern physics subjects such as superfluidity and hydrodynamics [3–6], solid state physics [7–9], electromagnetically induced transparency, slow light [10], and strong gravitational fields [11,12].

An atomic dilute gas BEC is formed when the wave packets of the atoms significantly overlap with each other. This condition is mathematically described as

\[ n\lambda_{dB}^3 \geq \zeta(3/2), \]  

in which \( n \) is the spatial atomic density, \( \lambda_{dB} \) is the deBroglie wavelength of the atoms, and \( \zeta \) is the Riemann zeta function [13]. The numerical value \( \zeta(3/2) \approx 2.612 \) is also referred to as the critical phase space density for BEC formation. Equation 1.1 can be rewritten in terms of temperature. We have

\[ T_c = \left( \frac{n}{\zeta(3/2)} \right)^{2/3} \frac{2\pi \hbar^2}{mk_B}, \]  

in which \( T_c \) is the critical temperature, \( \hbar \) is the reduced Planck constant, \( m \) is the atomic mass, and \( k_B \) is the Boltzmann constant.

In the mean field regime, the behavior of BECs is described by the Gross-Pitaevskii (GP) equation, which is the Schrödinger equation with a nonlinear contribution due to the interactions between the atoms. At ultra-low temperatures the contact interaction \( (V(r) \propto \delta(r)) \) remains a good approximation as long as the majority of the interactions between the atoms occur in the
s-wave scattering channel. The 3D Gross-Pitaevskii equation can be written as

\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \mathbf{r}^2} + V(\mathbf{r}) + \frac{4\pi \hbar^2 a_s}{m} |\psi(\mathbf{r})|^2 \right) \psi(\mathbf{r}) = \mu \psi(\mathbf{r}),
\]

(1.3)
in which \(V(\mathbf{r})\) is the external potential, \(a_s\) is the scattering length, and \(\mu\) is the chemical potential.

The first and the second terms on the left-hand side of Eq. 1.3 represent the kinetic and the potential energy contributions to the total energy of the system. These terms are also present in the Schrödinger equation. The third term on the left-hand side is the mean field energy due to the interaction between the atoms. Due to extremely low temperatures, a point contact potential \(\delta(\mathbf{r}_1 - \mathbf{r}_2)\) is a good approximation for the system, which introduces nonlinearity and brings lots of rich and exotic physics to the system such as the phonon modes, solitons, vortices, and shock waves.

Modifying each term on the left-hand side of the GP equation opens new doors for simulating various physical phenomena some of which are mentioned above. Engineering the dispersion relation (kinetic energy) is the primary topic of my dissertation. As a mainstream field of research, engineering the dispersion facilitates simulating Hamiltonians that are common in solid state physics. Spin-orbit coupling (SOC) is a robust method for modifying the dispersion as well as a common phenomenon in solid state physics that plays a major role in understanding various physical phenomena [14, 15] such as the spin Hall effect [16]. SOC can be achieved e.g. by Raman dressing between the atomic hyperfine states, coupling lattice bands using shaking or moving optical lattices [17, 18], or in general by coupling an internal degree of freedom to the translational degree of freedom. We present the experimental methods and results for engineering the dispersion using SOC via Raman dressing in Chapter 3 and optical lattice band coupling in Chapter 4.

Modifying the potential energy helps to simulate various systems such as lattices in solids and time-dependent potentials such as sweeping barriers for studying hydrodynamics. Dynamic optical lattices [19–21] and more recently arbitrary time varying light field projection [17, 22] have been employed expanding the horizons of the feasible experiments. Imaging techniques with single site
resolution capability have expanded our knowledge about microscopic correlations and entanglement of ultracold gasses in optical lattices [23–26].

Engineering the interactions between the atoms can be realized using Feshbach resonances in which the scattering length between the atoms can be significantly modified by carefully changing the magnetic field applied to the atoms [27]. Currently, finding, characterizing, and utilizing Feshbach resonances is a very active field of research in atomic and molecular physics [28].

I start this dissertation by providing information about the BEC machine and the BEC generation process in Chapter 2. The necessary steps for making BECs and performing the experiments are listed and illustrated. Modifications to the apparatus are presented in detail for future reference and a better understanding of the experiments presented in the following chapters.

In the context of SOC, the original proposal for realizing the SOC using Raman dressing was put forward by Higbie et al. [29], and it was first experimentally realized at NIST by Lin et al. [9]. The Hamiltonian of the SOC systems can be written such that a gauge field is present similar to the Hamiltonian for charged particles in an electromagnetic field. This allows us to induce synthetic electromagnetic fields that effectively interact with neutral atoms (e.g. see [30]). We study the interactions in a SOC BEC by measuring its excitation spectrum using Bragg spectroscopy at the beginning of Chapter 3. The significance of this work is due to the presence of a Roton-like mode in the spectrum that resembles the Roton mode in superfluid liquid Helium. We show that the Roton-like feature can be precisely tuned by modifying the Raman laser parameters. We provide experimental evidence that the mode can be excited by sweeping a barrier through the superfluid.

It is known that the Galilean invariance and parity are broken in a SOC BEC. This is demonstrated in the experiment by letting the condensate expand in one dimension by suddenly relaxing the trap in that direction. A clear asymmetry in the expansion is observed, which is in good agreement with the Gross-Pitaevskii (GP) equation predictions. Our study shows that the condensate partially occupies the states with negative effective mass and this causes the same part of the condensate to rapidly slow down although the mean field force is still present. It is also shown that a single band similar to the lower SOC band captures the physics of the problem despite the fact that
SOC dispersion has various spin states at different quasimomenta. Our observations pave the way toward understanding the SOC and negative effective mass hydrodynamics using a simple single band model and motivate more theoretical and experimental studies.

In Chapter 4, we engineer the dispersion by coupling $s$ and $p$ lattice bands using a weak moving lattice while the lattice bands are treated as pseudo-spin states. In contrast to the SOC with Raman dressing, which has the same dispersion relation for the pseudo-spins, the dispersions of the $s$ and $p$ lattice bands are inverted. It is known from topological insulators and superconductor physics that inverted band dispersions, together with SOC, play a critical role for topological properties of materials [31–33]. One outstanding feature of the coupling scheme employed in these experiments is the asymmetry of the effective dispersion, which exhibits a local minimum located at a finite quasimomentum similar to that of the Raman dressed SOC. These results open doors for exploring exotic phenomena in optical lattice systems, such as Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) phases [34, 35] and Majorana fermions [33], in which the $s$ and $p$ band pseudospins are highly desirable [36–39].

Chapter 5 of my dissertation reports on observing novel Bose gas solitonic states. Solitons exist as strongly nonlinear and macroscopic excitations of Bose-Einstein condensates and degenerate Fermi gasses [40,41]. So far, various solitonic states have been proposed and experimentally realized such as dark matter-wave solitons [6,42–44], bright matter-wave solitons [45–49], gap matter-wave solitons [50], and dark-bright (DB) solitons [51]. We report on the observation of dark-anti-dark (DAD) solitons, dark-bright-bright (DBB) solitons, and dark-dark-bright (DDB) solitons. DAD solitons have structural similarities to the recently proposed magnetic solitons [52]. However, there are differences because, in DAD solitons, the sum of the densities is not constrained to be equal to that of the ground state in contrast with the magnetic solitons. Because of the limited number of the experimentally realized solitons, the discovery of novel solitons is of great interest and it stimulates new theoretical and experimental studies in the field.
CHAPTER 2

OVERVIEW OF THE BEC MACHINE

2.1 INTRODUCTION

In this chapter, we provide some insight into the working principles and details of our BEC machine used for my dissertation experiments. This machine has a long history in our group since producing its first BEC in 2004. So far, it has produced a few hundred thousand BECs.

Here an overview of different components of the machine is given followed by a more detailed description of them later. These components are necessary for making BECs as well as performing the experiments. The main components of the machine are:

- Vacuum chamber
- Control system
- Magneto-Optical Trap (MOT)
- Track system
- Magnetic trap
- Chiller system
- Laser systems
- Crossed dipole trap
- Imaging systems
2.1.1 An introduction to BEC generation

Figure 1 schematically shows the layout of the main optical table setup (science table) where the BEC is generated. The laser beams shown in red on the right-hand side of the figure represent the MOT beams. Only one vertical MOT beam is shown in the figure since it overlaps with the other vertical beam. Note that the lasers used on the science table are generated on a separate optical table and are brought to the table by optical fibers. We use non-polarization-maintaining optical fibers for transporting the laser beams, and control the polarization using a $\lambda/4$ and a $\lambda/2$ flaps followed by a beam cube for polarization cleaning. This method has proven to have excellent stability while being cost effective.

The MOT is loaded for 10 s in the presence of Light Induced Atomic Desorption (LIAD) light. The procedure is followed by a compressed MOT, molasses, and optical pumping stage. The atoms are magnetically captured using the transfer coils (also used as MOT coils) shown with the orange loops (see Fig. 1). The transfer coils are mounted on a stable motorized track shown in Fig. 1. While captured with the transfer coils, the atoms are transferred approximately 80 cm across the vacuum chamber and the differential pumping stage to the magnetic trap shown on the left-hand side of the figure.

The atoms are then captured in the magnetic trap and evaporatively cooled using a series of RF sweeps until the BEC is formed. During normal operation, the atoms are cooled to degeneracy in the magnetic trap. The cooling sequence is different when an optical trap is required. In this case, the atoms are partially RF cooled inside the magnetic trap and then transferred to the optical trap for further cooling. The cooling in the crossed dipole trap occurs by slowly weakening the trap such that only the hotter atoms leave the trap and the remaining cooler atoms condense into a BEC.

After cooling, the experiment is performed with the BEC. Based on the nature of the experiments, a new setup around the magnetic trap or a new sequence program is required. After the experimental procedure is finished, we probe the BEC using absorption imaging. We will explain these steps in more detail later in this chapter.
Figure 1: Schematic of the science table layout. The MOT beams are shown in red; the dipole beam is presented in yellow, the Ti-sapphire laser, 1540nm, auxiliary beam pass, and the imaging beam are shown in green, orange, blue, and cyan, respectively.
2.2 BEC MACHINE COMPONENTS

2.2.1 Vacuum chamber

Schematic of the vacuum chamber is shown in Fig. 2. The $^{87}\text{Rb}$ atoms (or $^{40}\text{K}$ atoms for generating a Degenerate Fermi Gas) are released by the dispensers in the vacuum chamber. The pressure at this stage is $10^{-8} - 10^{-9}$ Torr. The differential pumping stage provides a much better vacuum at the Pyrex science cell providing long BEC lifetime (more than 1 min lifetime has been observed).

![Figure 2: Schematic of the vacuum chamber is presented.](image)

After its assembly, the vacuum chamber has been baked out for approximately two weeks to make sure that water and other contaminants are evaporated out of the system. During the bakeout, the chamber is being evacuated by a turbo pump (Pfeiffer model#:TMU-071YP) backed by an oil-free roughing pump. Then the ion pumps are activated while the turbo and the roughing pump are segregated from the system. The differential pumping is achieved by two ion pumps positioned on either side of the differential pumping stage which simply is a narrow vacuum tube. The pumping speeds of the ion pumps are 20 Liters/s and 50 Liters/s before and after the differential pumping stage.

2.2.2 Control system

Precise timing up to tens of micro seconds is crucial to produce the BEC and perform experiments in a reproducible way. DIO64 cards (Viewpoint systems) are used providing 128 bits of data with internal timing control which allows for the time resolution of 1 $\mu$s. A labview interface communicates with the DIO boards and assigns values to the outputs in the programmed time
sequence. The Labview interface is programmed using a C++ script in the MFC (Microsoft Visual C++) environment. Using a C++ program reduces complexity and it is a more powerful method compared with a labview graphic user interface. For example, the C++ program allows us to write different sections for different experiments which are controlled by boolean flags while keeping the common part of the program the same. Also, we can easily control the flow of the commands by manipulating the execution time of each command. As an example, assume we need to apply a magnetic field to the atoms seconds before the BEC is formed and we need to change the amount of the field in a series of experiments to study the effect. If a GUI was used, every time a parameter had to be changed, we had to scroll to that specific section of the program and graphically change the parameters. In contrast, in the C++ program, we can easily keep the command that turns on the magnetic field at an arbitrary position but with a retarded time parameter.

32 TTL outputs of the first DIO64 card are buffered and electronically isolated using opto-couplers for digital outputs such as enablers or triggers. The remaining channels are preserved for analog outputs. Three bits are reserved for addressing the DAC channels allowing for $2^3 = 8$ multiplexer channel combinations. Each channel uses a 16-bit digital to analog output. One TTL is used for triggering the analog outputs and the remaining 12-bits are used for an 11-bit DAC output with the last bit used for triggering the output. The second DIO board is broken down to 44 TTL outputs plus eight 16-bit DACs plus three address bits and a trigger bit.

The DS345 and AFG3101 frequency generators as well as the track system (Parker - SM3440D) are programmed via GPIB and triggered via the DIO boards for timing consistency.

All these TTL and DAC outputs precisely control the laser intensities, currents, triggers, shutters, and switches. We use the second computer to acquire the experimental images from the cameras.
The MOT is generated using six red detuned counter-propagating laser beams intersecting at the center of an anti-Helmholtz coil assembly. Each beam approximately carries 30 $mW$ of power and has a diameter of approximately 1.5”. The MOT lasers are $\approx 14$ $MHz$ red-detuned with respect to the $5S_{1/2} F = 2$ to $5P_{3/2} F = 3$ line (see Fig. 4).
Figure 4: Relevant energy structure of $^{87}$Rb. The red and the blue arrows on the left-hand side of the figure correspond to the D1 and D2 transitions, respectively [1].

The anti-Helmholtz coils are water cooled and mounted on a motorized track for transferring the atoms from the MOT to the science cell for evaporative cooling. The magnetic field gradient induced by the coils is approximately $4 \text{ G/cm}$ with current $\approx 9.6 \text{ A}$ passing through the wires. The atoms are accumulated for 10 s assisted by the LIAD (Light Induced Atomic Desorption) lights with a wavelength of approximately 400 nm. The LIAD lights quickly increase the MOT chamber background pressure for larger MOT loads and abruptly reduce the pressure after being turned off to minimize collisions of the cold atoms with the room-temperature atoms during the transfer. We roughly gain a factor of five in the MOT atom number using the LIAD lights.

The steps that occur in the MOT chamber are as follows:

1. MOT loading: Includes the LIAD assisted accumulation of atoms

2. MOT compression: Sudden increase in the magnetic field gradient to squeeze the cloud and avoid heating while magnetically capturing the atoms.

3. Molasses cooling: Expansion of atoms without the presence of the magnetic fields for further
4. Optical pumping: Depending on the atomic state employed for the experiment, the proper optical pumping is applied. For example, to make a BEC in the $|2, 2\rangle$ state, a small bias field is applied in the MOT, and then the optical pumping laser light is turned on. The optical pumping light is a circularly polarized laser light that is resonant with the $F = 2 \rightarrow F = 2'$ transition. For the case of atoms in the $|1, -1\rangle$ state, we stop the repump laser for 5 ms after the compressed MOT period. This makes the atoms fall in the $F = 1$ state.

5. Capture the atoms in the quadruple trap.

It is worth noting that in the case of optical pumping to the $|2, 2\rangle$ state more atoms will be magnetically trapped versus in the $|1, -1\rangle$ case in which the atoms present in the $|1, 0\rangle$ and $|1, +1\rangle$ states are lost since they can not be trapped at a magnetic minimum.

In the following, we give a short overview of the MOT laser and the repump laser setups:

**MOT laser:** As mentioned previously, the MOT laser system is needed for cooling the atoms down to sub-mK temperatures. To achieve this, the laser has to have enough stability and power. The stability is achieved using a PID loop constantly locking the ECDL to the $^{87}$Rb lines. The power is provided via a tapered amplifier (TPA-780-0500) generating 450 mW seeded by the $\approx 10$ mW ECDL. The beam is transported to the science table via a single mode non-polarization maintaining fiber. The polarization is then adjusted using two $\lambda/2$ and $\lambda/4$ flaps.

In this setup the spectroscopy beam is locked to the cross-over peak between the $F' = 2$ and $F' = 3$ of the $D2$ line. The spectroscopy beam is branched off of the first negative order of the 120 MHz AOM. Therefore, the beam seeding the tapered amplifier is 120 MHz larger in frequency than the spectroscopy beam. Since the splitting between $F = 2'$ and $F = 3'$ in $D2$ transitions is $\approx 266.65$ MHz, and the crossover peak is in the middle of these transitions, the MOT beam is $\approx 13.3$ MHz red-detuned with respect to the MOT transition (see Fig. 4). Figure 5 shows the MOT laser layout of our setup.
Repump laser: Since some of the atoms in the MOT transition end up in the $F = 1$ manifold of the $5^2 S_{1/2}$, a repump laser is required to bring these atoms back to the MOT cycle. The leaking transition has smaller probability than the MOT transition. Thus only a few milliwatts of repump laser power is sufficient for reactivating the “fallen” atoms.

2.2.4 Track system

After the optical pumping phase, the atoms are caught in the quadruple trap by quickly ramping up the current in the MOT coils. The magnetic gradient is approximately 150 $G/cm$ as $387$ $A$
of current runs through the coils. Then the captured atoms are transferred by a motorized track (model#SM3440D) for about 0.8 m across the differential pumping stage in 2.5 s to the Pyrex science cell (science cell in short) for further cooling. The atoms are then captured in the magnetic trap surrounding the science cell. The process is orchestrated by ramping down the current in the MOT coils and simultaneously ramping up the current in the magnetic trap coils. Good alignment between the center of MOT coils and the minimum magnetic field of the magnetic trap is essential for high transfer efficiency.

2.2.5 Magnetic trap

The schematic of the magnetic trap and the science cell is presented in Fig. 7(a). The Ioffe coils provide the radial confinement (along the $y$ and $z$ directions), the pinch coils provide the axial confinement (along the $x$ direction), and the bias coils provide a homogeneous magnetic field in the axial direction. The Ioffe coils are connected in anti-Helmholtz configuration. The pinch coils are in Helmholtz configuration but situated relatively far from each other to generate a non-zero magnetic field minimum with sufficient curvature to provide axial confinement. The bias magnetic field and the pinch magnetic fields are in opposite directions.

After the atoms are captured in the magnetic field, the Radio Frequency (RF) evaporation process starts. Hot atoms are evaporated away from the trap and being pumped out. This is done using an RF ramp generator, an RF amplifier, and an RF coil mounted close to the science cell. This setup is schematically presented in Fig. 7(b). Depending on which trap is used, the RF evaporation sequence is modified. For example, if we need the BEC to form in the magnetic trap, we apply the evaporation until the BEC is formed. In contrast, if we need a BEC in the dipole trap, we evaporate the atomic cloud to approximately 200 kHz above the normal RF endpoint, and then we slowly sag the position of the atoms by weakening the magnetic trap and capturing them by the dipole trap. Once the atoms are captured by the dipole beam, we slowly reduce the laser intensity until a BEC is formed.

To excite the atoms to the $F = 2$ state, a microwave antenna is placed near the science cell.
The MW signal is created using a constant frequency MW generator mixed with an RF generator for frequency modulation. The block diagram of the MW generation setup is shown in Fig. 7(c).

Figure 7: (a) Schematic of the magnetic trap. (b) and (c) represent the schematics of the Radio Frequency (RF) and the Microwave (MW) circuits.

**Magnetic field configuration in the magnetic trap**

As mentioned, the magnetic trap provides a non-zero magnetic field minimum in space, which provides a potential energy minimum to the atoms in specific Zeeman states. These atomic states such as $|F = 1, m_F = -1\rangle$, $|F = 2, m_F = 2\rangle$, and $|F = 2, m_F = +1\rangle$ are the ones that experience an increase in the Zeeman energy level by increasing the magnetic field. Therefore, naturally, they experience a force pushing them towards the minimum magnetic field at approximately the center of the trap. The reason it is only approximately the center of the trap is that Earth’s gravity adds an extra force in the vertical direction. Also, the Earth’s magnetic field and other small perturbations slightly modify the trap geometry.

The magnetic field strength in the radial direction (perpendicular to the $x$ axis), which is generated by the Ioffe coils, increases linearly with distance (see Fig. 8 (a)). The magnetic field in the axial direction $x$ is plotted in Fig. 8 (b). Together with the Ioffe coils, the Pinch and the Bias coils form a magnetic field minimum illustrated in Fig. 8 (c). By accounting for Earth’s gravity in the potential, the minimum shifts slightly downward. The amount of the vertical shift
can be controlled by the magnitude of the Bias magnetic field. We exploit this feature when we want to load the optical trap from the magnetic trap. We slightly sag the magnetic trap down (approximately 400 µm) towards the dipole beam and transfer the atoms from the magnetic trap into the dipole beam.

![Diagram](image)

Figure 8: (a) Magnetic field along the radial direction (in the $y - z$ plane) at the center of the trap. (b) Magnetic field along the longitudinal direction for different bias magnetic fields (dashed lines). (c) Potential energy due to the magnetic field around the trap center. (d) Trap potential energy for a trappable atomic state including the earth gravitational pull.

### 2.2.6 Chiller system

With currents as high as $\approx 400 \, A$ through the coils during transferring or magnetically trapping the atoms, a lot of heat ($\approx 4000 \, W$) is dissipated in the wires. This heat can quickly cause damage, extreme expansions, and contractions in each cycle. To keep the temperatures low in a reproducible fashion, we use a servoed heat exchanger that keeps the temperature of the pressurized water that runs through the coils constant.

We use distilled water for our closed cycle cooling mechanism. Distilled water minimizes minerals deposition in the wires. Flow switches measure the water flow in individual branches. The outputs of the flow switches are connected in parallel. If a flow switch detects no or small flow, it shorts its output wire leads and therefore deactivates the power supply.

The temperature of the water reservoir is kept at approximately $20^\circ C$ using a closed cycle heat exchanger and an actuated valve (Siemens model# SQS65.5U) that controls the flow of the building chilled water through the heat exchanger. The temperature of the building chilled water is $\approx 9^\circ C$. The distilled water is then pumped through the coils. The pressure of the water is approximately 150 $psi$ immediately after the pump to ensure enough flow throughout the circuit. In each cycle, the temperature of the magnetic trap coils increase from $20^\circ C$ to $25^\circ C$ while the atoms are trapped.
in the magnetic trap. The temperature of the coils is continuously monitored via thermocouples. A warning will sound if the temperatures increase above 40 °C.

It is worth noting that temperature fluctuations of the coils have a direct impact on the evaporation process due to slight changes in the magnetic field.

Figure 9: Schematic of the chiller system.

Figure 9 shows a schematic of the chiller system. This design is an evolved version of the previous system in which the actuator controlled the temperature of the reservoir through a bypass. The new design has proven to be more reliable and more stable. The downside of this configuration is that the building chilled water has a lot of residuals and it may damage the actuated valve.

2.2.7 Laser setups

We use different laser beams for different purposes when experimenting with $^{87}\text{Rb}$ atoms. We mention these laser setups in this section and provide more details about them. The most important
laser setups used in our experiments are as follows:

- 1064 nm laser: Dipole trapping and optional lattice generation
- Ti-Sapphire laser: Wide variety of wavelengths for lattice generation, Raman coupling, etc.
- 1540 nm laser: Lattice, and superlattice generation.

**1064 nm laser**

The 1064 nm CW, single mode, single frequency, laser (model# YLR-20-1064-LP-SF) is used for different purposes in our experiments. Most importantly this laser is used in the crossed dipole trap discussed in 2.2.8. We also used the 1064 nm laser for generating lattices using two auxiliary output couplers on the optical table (see Fig. 1, the blue beams). The schematics of the 1064 nm laser setup is presented in Fig. 10.

![Figure 10: Schematics of the 1064 nm laser setup.](image-url)
The lattice beams are branched off from the third beam cube after the output coupler and again are split into two beams and are transferred to the science table.

**Ti-Sapphire laser**

There are normally two Ti-Sapphire laser (M² model# SolsTis) generated beams used in our experiments for lattice generation or Raman dressing. For generating static and moving lattices, we optically combine a third beam with both input couplers to form a static lattice. This is shown as the beam deflected from the 100 MHz AOM in Fig. 11. To avoid interference between the moving and the static lattice, we use the positive order of the 100 MHz AOM while the moving lattice utilizes the negative orders of the 80 MHz AOMs.

The laser allows locking the wavelength using two mechanisms, namely cavity, and etalon. Both locks are required to be able to lock the laser to a single mode. This is especially useful when we are doing spin-orbit-coupling as the detuning is sensitive to the laser wavelength, and mode jumps can cause drifting of the resonance. At least $\approx 4.5$ W of pump power is required to be able to lock the laser.
Figure 11: The schematics of the Ti-Sapphire laser setup. The system consists of two input couplers fed by independent AOMs, and a third beam is combined to both serving as an optional standing lattice.
1540 nm laser

A 1540 nm (NP Photonics) Rock laser seeds a fiber amplifier (IPG model# EAR-LP-SF) to provide two beams to generate a lattice. This lattice has been used for Bragg spectroscopy of the Roton minimum in the excitation spectrum of spin-orbit-coupled BEC.

![Figure 12: Schematic of the 1540 nm laser setup.](image)

2.2.8 Crossed dipole trap

In many cases spin dependent measurements are required for our experiments. For this purpose, we use a crossed optical dipole trap, which consists of a vertical beam intersecting a horizontal beam known as the "Dipole Beam". The dipole beam provides support against gravity while the vertical beam provides axial confinement. Figure 13 shows the schematic of the crossed dipole trap beam paths in our setup. We use a single mode, single frequency, CW, 1064 nm (IPG model# YLR-20-1064-LP-SF), which is far detuned from the atomic transitions and therefore provides long BEC lifetimes.

The atoms in the crossed dipole trap are loaded by slowly sagging the magnetic trap to the position of the dipole beam (approximately 400 µm below the magnetic trap) and ramping up the dipole beam while ramping down the magnetic trap. The vertical beam is later ramped on during the evaporative cooling inside the dipole beam. The dipole beam has a beam waist of $\approx 26 \mu m$ while the vertical beam has a beam waist of $\approx 120 \mu m$. 
Figure 13: Schematic of the Crossed Dipole Trap. (a) Schematic of the vertical beam and the vertical imaging beam. Pixelfly camera is an auxiliary camera used for vertically imaging the condensate. (b) Schematic of the dipole beam.

Based on the application, different trapping frequencies are used. For most of the experiments we use a trapping geometry with typical trapping frequencies \((\omega_x, \omega_y, \omega_z) = 2\pi(25, 170, 150)\) Hz. For an elongated condensate, the vertical beam can be turned off. The focusing of the dipole beam provides some weak trapping along the \(x\) direction. This trapping geometry has frequencies \((\omega_x, \omega_y, \omega_z) = 2\pi(1.4, 170, 172)\) Hz.
Figure 14: (a) Illustration of the crossed dipole trap beam configuration. (b) Shows the calculated equipotential surfaces close to the center of the crossed dipole trap. The potentials due to the light field from the outermost to the innermost layers are 60, 40, 20, and 5 kHz respectively. (c) The cross section of the potential contours with the presence of gravity. (d) 1D potential cross sections along the $x$, $y$, and $z$ directions with the corresponding trapping frequencies. The difference between the measured trapping frequencies and the calculated ones can be understood by considering the fact that the conditions perfect in the simulations. However, in reality, there are imperfections both in the lens and the laser beam that can cause the difference between the simulation and the experiment.

Figure 14 illustrates the schematic and the typical simulation results for the crossed dipole trap.

Figure 14 (b) shows the partial waves simulation results for a crossed dipole trap. In the simulations
the dipole beam has an initial beam waist of 3.1 mm, 250 mm focusing lens, and 24 mW of laser power and the vertical beam has an initial beam waist of 1.4 mm, 185 mW of power, and a 500 mm focusing lens.

In the simulations code, we start with a source plane with an arbitrary intensity and phase pattern (Gaussian beams with flat phase front are used for this simulation). The propagation direction is perpendicular to the propagation plane. After passing through a focusing lens, the beam is focused at the center of the lab coordinate system. The intensities due to different beams are calculated and added to find the final intensity. For more information about the simulation methods see Sec. 2.8.

2.2.9 Imaging system

We use absorption imaging techniques using two different transitions namely $F = 1$ to $F = 1'$ and $F = 2$ to $F = 3'$ depending on the required contrast.

![Figure 15: Schematics of absorption imaging technique. The atomic cloud casts a shadow on the CCD.](image)

The imaging laser setup uses a New Focus Vortex ECDL Laser. The reason we use this laser instead of our homemade model is its tunability and large mode-hop free range as we need different transitions for imaging. For in-trap images we normally use the $F = 1$ to $F = 1'$ transition while for ToF images, we use $F = 2$ to $F' = 3$ transition for more contrast as it is a cycling transition. This increases the imaging contrast while the cloud is very dilute. Two counter-acting AOMs are utilized to enable fast switching of the beam. The imaging transitions can be found in Fig. 4. A schematic of the imaging laser setup is shown in Fig. 16.
We have used two imaging axes to image the condensate in our experiments. Figure 17(a) and 17(b) show the schematics of the imaging configurations. The main imaging axes shown in Fig. 17(a) is along the vacuum system. This imaging beam (shown in cyan) enters the vacuum chamber through the window flange (also shown in cyan in Fig.1). The beam is targeted at the Princeton Instruments camera (PIXIS 1024BR) and the vertical imaging axis is aligned with the PixelFly camera (QE).
Figure 17: Schematics of the imaging systems. (a) Shows the main imaging axis utilizing the PI camera. (b) Shows the vertical imaging axis utilizing the PixelFly camera.

**PI (Princeton Instruments) camera imaging system**

The CCD of the PI camera is back illuminated, held under vacuum and at $-70^\circ C$ for better signal to noise. Each pixel is $13.4\,\mu m \times 13.4\,\mu m$. The quantum efficiency of the sensor at 780 nm is $\approx 95\%$. The camera is mounted on a vertical translation stage to align for different ToF ranges.

Since the image of the microscope is inverted, lower camera position is focused for shorter ToF images (typically 0 ms to 8 ms for the magnetic trap) while the higher camera position is focused for longer ToF images (typically 9 ms to 16 ms for the optical trap). Also, depending on the trap we are using, these numbers change since the optical trap is positioned in a sagged position with respect to the magnetic trap.

The microscope utilizes two lenses: 1. GRADIUM (Edmund Optics) lens (gradient index of refraction for better optical correction) $f = 4$ cm. 2. two inch diameter achromat $f = 70$ cm.
This setup has a magnification such that each image pixel corresponds to $0.766 \times 0.766 \, \mu m^2$ at the position of the atoms. The imaging system uses approximately 250 $\mu W$ applied for 10 $\mu s$ to generate the shadow frame.

**PixelFly camera imaging system**

This imaging setup aims at taking images of the BEC in the vertical direction ($z$ direction). We use this imaging axis mostly for measuring the trapping frequency along the $y$ axis (the machine axis). It allows us to go to longer ToF times since in this case we are not limited by the PI camera’s translation stage. The PixelFly camera has pixels of $6.45 \, \mu m \times 6.45 \, \mu m$, but it is not actively cooled, and it is not held under vacuum. However, this camera can capture very high temporal resolution data by an overclocked vertical shift in the pixel rows. The microscope for this setup consists of a $f = 100 \, mm$ achromat lens as the objective followed by an achromat lens with $f = 250 \, mm$. Each pixel in this imaging system corresponds to 0.447 $\mu m$. Similar to the previous case, we apply the imaging beam for 10 $\mu s$ to form the shadow frame.

**Imaging procedures**

Depending on the experiment, we use different imaging procedures to capture the relevant features of the experiment. For example, if ToF images with higher contrast are needed, we use the $F = 2$ to $F' = 3$ atomic transition and if in trap images with dense clouds are required, the $F = 1$ to $F' = 1$ atomic transition is utilized to avoid saturation. We will revisit the imaging saturation subject in the next subsection.

In many cases, we would like to have mixtures of different hyperfine states or Zeeman states in the condensate. In these cases, we need to distinguish between these different states in the images. We use two different procedures depending on the situation:

- **Stern-Gerlach imaging procedure**: A vertical magnetic field gradient is applied for 3 ms to spatially separate different Zeeman states with different magnetic moments. The procedure is followed by a microwave adiabatic rapid passage (ARP) to transfer the atoms from the
$F = 1$ to $F = 2$ state for absorption imaging using the $F = 2$ to $F = 3'$ atomic transition.

- **Dual imaging procedure:** In some experiments, we have mixtures of $F = 1$ and $F = 2$ hyperfine manifolds, and therefore one needs to apply two different imaging beams each being resonant with one transition. That would cause unnecessary complications to the setup. To overcome this issue we expose the camera twice with the $F = 2$ to $F' = 3$ resonant beam, but between the two exposures a microwave ARP is applied to transfer the remaining atoms from the $F = 1$ state to the $F = 2$ state. During the 1 ms ARP the remaining atoms in $F = 1$ state fall further down and therefore appear spatially separated from the atoms that were originally in the $F = 2$ state. The only downside of this imaging method is that some atoms that are imaged in the first exposure manage to decay to $F = 1$ state and contaminate the second exposure. We estimate this contamination to be approximately 35% of the atomic population originally occupying the $F = 2$ state.

Each experimental image consists of three frames:

- Shadow frame with intensity pattern $I_{\text{shadow}}$
- Light frame with intensity pattern $I_{\text{light}}$
- Dark frame with intensity pattern $I_{\text{dark}}$

The shadow frame contains the shadow of the atoms; the light frame has the imaging laser beam profile, and the dark frame has information about the baseline to cancel out the effect of stray light, etc [53]. The Beer-Lambert law, which relates the intensity of light as it travels through the medium, states:

$$I(x, y) = I_0 e^{-\int \sigma_{\text{abs}} n(x, y, z) dz}, \quad (2.1)$$

where $I_0$ is the imaging beam intensity, $I$ is the intensity of light as it travels through the medium, $\sigma_{\text{abs}}$ is the absorption cross section, and $n(x, y, z)$ is the density [53]. The absorption cross section can be written as

$$\sigma_{\text{abs}} = \frac{\hbar \omega \Gamma}{2I_{\text{SAT}} \left( 1 + 4 \left( \frac{\Delta \Gamma}{\Gamma} \right)^2 + \frac{I_0}{I_{\text{SAT}}} \right)}, \quad (2.2)$$
where $\Delta$ is the detuning of the laser beam from the atomic transition, $\Gamma$ is the natural linewidth and $I_{SAT}$ is the saturation intensity of the transition [13]. We define Optical Density (OD) as

$$OD = \int \sigma_{abs} n(x, y, z) dz,$$  \hspace{1cm} (2.3)

thus, OD can be obtained using

$$OD = \ln \frac{I_0}{I},$$ \hspace{1cm} (2.4)

in which $I$ and $I_0$ are defined in Eq. 2.1. However, in both the shadow and the light frame we have some stray light leaking into the camera. To take that into account we subtract $I_{dark}$ from both the light and the shadow intensities,

$$OD = \ln \frac{I_{light} - I_{dark}}{I_{shadow} - I_{dark}}.$$ \hspace{1cm} (2.5)

In Eq. 2.5, $I_{light} - I_{dark}$ and $I_{shadow} - I_{dark}$ correspond to $I_0$ and $I$ in Eq. 2.1, respectively.

In reality, extra care must be taken since the CCD receives some stray light from the environment, and it might get saturated. The stray light is usually transparent to the atomic transitions. After considering these situations we have:

$$OD_{real} = \ln \left( \frac{I_{in} - I_{bg}}{I_{out} - I_{bg}} \right) \left( 1 + \frac{I_{in} - I_{bg}}{I_{sat}} \right),$$ \hspace{1cm} (2.6)

in which $I_{in}$ is the total incoming intensity without the atoms, $I_{bg}$ is the total non-resonant intensity, $I_{sat}$ is the resonant light saturation intensity, and $I_{out}$ is the total measured intensity in the presence of the atoms. These quantities are measured during the three frame acquisition sequence.

### 2.2.10 Atom number calibration

At this stage, we have the optical density, which is defined in Eq. 2.3. We still need to calibrate the OD with the actual atom number. There are still a few steps that need to be taken to estimate the atom number based on the OD. First, we should estimate the imaging magnification. Second,
using the imaging magnification we should find the size of the BEC, and together with the trapping frequencies, we can estimate the atom number using the Thomas-Fermi (TF) approximation.

**Imaging magnification**

We estimate the imaging magnification by letting the atoms free fall and calculating the acceleration using the images. Since we know the real acceleration of the atoms, we can use that information to estimate the imaging magnification. Figure 18 shows the ToF data (a) and absorption images (b) of a BEC falling freely in Earth’s gravity. Starting with a BEC in the magnetic trap, we let the BEC freely fall by switching off the trap.

\[
Y = A + B_1X + B_2X^2
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>172.51888</td>
<td>2.55511</td>
</tr>
<tr>
<td>B_1</td>
<td>-0.94131</td>
<td>1.18879</td>
</tr>
<tr>
<td>B_2</td>
<td>6.40399</td>
<td>0.1145</td>
</tr>
</tbody>
</table>

The estimated acceleration is \(6.40 \pm 0.11 \text{ (pixels/m}^2\text{)}\) while the gravitational acceleration on Earth is \(9.81 \text{ (\mu m/m}^2\text{)}\). Therefore our magnification factor is \(0.766 \pm 0.02 \text{ (\mu m/pixels)}\). It is worth noting that this value might slightly change every time we refocus the camera since the exact position of the camera may not exactly coincide with the image focal plane.

**Trapping frequency**

Knowing the trapping frequencies is also necessary to estimate the atom number. We measure the trapping frequencies by slightly shifting the trap using a magnetic field gradient or by changing the
dipole trap intensity and suddenly bringing the trap back to its original configuration. This causes the condensate to oscillate in the trap revealing the trapping frequencies. The momentum of the condensate can be measured by measuring its position in ToF images. The oscillation frequency can be measured by acquiring a sequence of these measurements for different wait times.

![Figure 19: Coils around the science cell. In this figure the Ioffe coils are not visible.](image)

To excite the condensate along the axial direction ($x$–axis), we use one of the little bias coils to generate the gradient. Also to cause an excitation along the vacuum chamber ($y$–axis), we use a small coil (Machine axis coil) to create a gradient along that direction. Figure 19 shows the little bias coils as well and the machine axis coil and the bias coils. The reason we can not use the bias coils for exciting the BEC along the $x$ direction is that they are hardwired in the Helmholtz configuration and can not be driven individually.

For the atom number calibration, we use the crossed dipole trap since it is convenient to generate excitations in this trap magnetically and optically. The trapping frequencies may change slightly from day to day depending on the alignment between the crossed dipole beams. Figure 20 shows an example for a trapping frequency measurement along the $x$ axis. The sinusoidal fit gives 24.6 Hz for that particular trap configuration.
Figure 20: Axial trapping frequency measured for atom number calibration. Other trapping frequencies are measured in a similar fashion.

**Thomas-Fermi approximation**

In the time independent Gross-Pitaevskii equation, the kinetic part becomes relatively small and therefore only the potential and the nonlinear terms become important. We have,

\[
\frac{p^2}{2m} \psi(\vec{r}) + V(\vec{r}) \psi(\vec{r}) + g|\psi(\vec{r})|^2 \psi(\vec{r}) = \mu \psi(\vec{r}),
\]

(2.7)

In which, \( \psi \) is the wave function, \( g \) is the interaction energy scalar and \( \mu \) is the chemical potential. Thus by simplifying Eq. 2.7 we have,

\[
V(\vec{r}) + gn(\vec{r}) = \mu,
\]

(2.8)

where \( n(\vec{r}) = |\psi(\vec{r})|^2 \) is the density. Therefore the atom number in the condensate can be approximated as

\[
N = \frac{1}{g} \int (\mu - V(\vec{r})) d^3 r.
\]

(2.9)

Equation 2.9 can be simplified for a harmonic trap with trapping frequencies \((\omega_x, \omega_y, \omega_z)\) as,

\[
N = \left( \frac{2\mu}{\hbar \omega} \right)^{\frac{3}{2}} \frac{a_{ho}}{15a_s},
\]

(2.10)
in which $\bar{\omega} = \sqrt[3]{\omega_x \omega_y \omega_z}$, $a_{ho} = \sqrt{\frac{\hbar}{m\omega}}$. Note that in Eq.2.10, $\mu$ (the chemical potential), can be calculated since the TF radius is known from the experiment:

$$\mu = \frac{1}{2} m \omega_i^2 R_i^2 \quad (2.11)$$

in which $i = x, y, z$. Note that the chemical potential is a scalar quantity and is direction insensitive. Therefore, it is the same for any choice of $i = x, y, z$. Experimentally, we choose the direction that has the minimum uncertainty.

**Imaging calibration**

Now that the atom number is estimated using the TF approximation, we need to relate the atom number to the optical density or the camera pixel counts. In this procedure we let the BEC expand, take ToF absorption images of the condensate, and integrate the total pixel intensity. The integration (summation) is done using the Winview software.

In this process, care must be taken such that the images do not saturate. Saturation occurs when light cannot make it through the cloud to the CCD. This typically happens for the ToF images taken using the $F = 2$ to $F = 3'$ transition since this is a cycling transition. This can also occur when in-trap images are taken using the $F = 1$ to $F = 1'$ transition. To avoid saturation, we measure the pixel counts of the BEC with known atom number in ToF using the $F = 1$ to $F = 1'$ transition and separately find the ratio between the pixel counts between $F = 1 \rightarrow F = 1'$ and $F = 2 \rightarrow F = 3'$ using a dilute (extended) cloud. To produce dilute clouds we prepare the BEC in the dipole beam instead of the crossed dipole trap.

### 2.3 Dual Manifold Spin-Orbit Coupling for $^{87}$Rb

A magnetic bias field plays a crucial role in SOC [9]. Traditionally, the active levels in the spin-orbit coupling have been the Zeeman sublevels of the $F = 1$ manifold, which experience an energy shift of approximately 700 kHz/G. This means that to be able to control the Zeeman splitting to better
than 100 Hz we need to control the magnetic field to better than 140 \( \mu \)G. Considering the magnetic field noise around the chamber, this could be a challenging task, and it would save a lot of time and effort if it can be avoided.

Due to the equal and opposite \( g \)-factors of \( F = 1 \) and \( F = 2 \) manifolds, Zeeman sublevels with opposite signs but equal magnitude \( m_F \) number do not shift relative to each other to the first order in magnetic field. This holds as long as the quadratic Zeeman shift due to the magnetic field fluctuations remains small.

### 2.3.1 Experimental Setup

The dual manifold SOC, which is a three-photon process, requires a simultaneous presence of the microwave (MW) signal, as well as the Raman beams all at the same time. However, both the Radio Frequency (RF) and the MW antenna and their sources are used during the experiment for evaporation and imaging. Therefore we need to modify the RF and microwave circuit such that we can apply the appropriate signals upon request. The complication arises because the SRS DS345 function generators can not generate frequencies above 30 MHz, and therefore the AFG signal generator, which is usually used for evaporative cooling, needs to be used for MW generation.

The schematic of the RF and MW circuits is presented in Fig. 21.
The three-photon SOC level scheme is presented in Fig. 22 (a) where the microwave is approximately 200 kHz detuned above $|2,0\rangle$ state. The reason for this detuning is to avoid populating the $|2,0\rangle$ state during the process.

We use the RF pulse simultaneously with the MW for locating the $|2,-1\rangle$ resonance before we use the Raman beams. The schematics of this two-photon transition is shown in Fig. 22 (b) where instead of the two-photon Raman transition an RF pulse is used.

Figure 21: RF and Microwave circuits schematics.
2.4 BILAYER SYSTEM

By utilizing a Hermite-Gauss beam, we can generate a bilayer spin-orbit coupling or a general bilayer attractive or repulsive beam to induce a bilayer profile on the condensate.

Figure 22: (a) Schematic of the three-photon level scheme. (b) Schematic of the MW + RF spectroscopy level scheme.
Figure 23: Schematics of the bilayer setup.

We utilize an uncoated, 1” in diameter, N-BK7 Broadband Precision Window, with the thickness of $d = 5 \ mm$, from Thorlabs to prepare the phase plate. Half of the window is coated with a thin film to generate a relative phase shift. The extra phase that the light picks up when traveling through the phase shifting film is

$$\phi_{\text{material}} = \frac{2\pi}{\lambda} nx,$$  \hspace{1cm} (2.12)

and the phase that the light picks up in the similar length of air is

$$\phi_{\text{air}} = \frac{2\pi}{\lambda} x.$$  \hspace{1cm} (2.13)
Thus the phase difference is described as

\[
\Delta \phi_{\text{total}} = \frac{2\pi}{\lambda} (n - 1)x,
\]

in which \( \lambda \) is the laser wavelength 789 nm. Thus,

\[
x = \frac{\Delta \phi_{\text{total}} \lambda}{2\pi (n - 1)}
\]

where \( \Delta \phi_{\text{total}} = \pi \), \( \lambda = 789 \) nm, and \( n = 1.46 \). If a film with index of refraction of 1.46 is deposited, the height \( x \) needs to be 857.6 nm for a phase shift of \( \pi \). Note that in all of our calculations we neglected all other layers such as the anti-reflection coating as they have the same thickness everywhere and accumulate the same phase shift. After the phase shifting layer is applied, the window is AR coated.

Finally, the output of the phaseplate is then focused at the position of the atoms using a 15 cm achromat lens.

### 2.4.1 Alternative design for phaseplate

We can use a transparent rotatable slab to generate an arbitrary relative phase depending on the rotation angle of the slab and its index of refraction. A schematic of the beam paths is shown in Fig. 25 (a).
In Fig. 25 (b) the solid beam represents the actual beam path and the dashed line represents the beam missing the slab.

Assuming the index of refraction and the thickness of the slab are \( n \) and \( d \), respectively, the light path length \( D_n \) due to the slab can be written as

\[
D_n = \frac{nd}{\cos (\theta')} + d \tan (\theta) \frac{\sin (\theta - \theta')}{\cos (\theta')}, \tag{2.16}
\]

in which \( \theta \) is the tilting angle of the slab with respect to the beam direction and \( \theta' \) the the refracted beam angle inside the slab. The first term is the path length due to the light traveling inside the slab and the second term is the path difference due to the parallel shift.

The corresponding light path length in air is

\[
D_{air} = \frac{d}{\cos (\theta)}. \tag{2.17}
\]

Therefore, the phase difference between the beam passing through the slab and the air is

\[
\Delta \Phi = \frac{2\pi}{\lambda} (D_{air} - D_n), \tag{2.18}
\]

in which \( \lambda \) is the wavelength of the light. The angle dependence of the phase difference is shown in

---

Figure 25: Adjustable phaseplate schematic. (a) Dimensions and geometry of the problem. (b) The path difference between a beam passing through the slab (solid line) and a beam path in absence of the slab (dashed line).
Fig. 26 (a) for a 3 mm thick slab with index of refraction $n = 1.5$ and wavelength 788 nm. Figure 26 (b) shows our designed slab holder. The mount is cut in the middle except for a narrow hinge structure on the right. The screw holes are for adjusting the angle between the front and the slab angle. The window must be half covered with the slab (the slab not shown in the figure) such that half of the light passes through the slab and half passes through the air.

Figure 26: Adjustable phase plate. (a) The angle dependence of a 3 mm thick slab with an index of refraction $n = 1.5$, and the wavelength of 788 nm. (b) Our tested variable angle phase plate. The screw holes A, B, and C are for adjusting the angle between the slab holder (shown in the front) and the base (shown in the back). The window D holds the slab such that half of the window is left uncovered.

2.5 COMBINED LATTICES WITH THE SAME PERIODICITY

We use a weak running optical lattice to couple the Bloch bands created by a strong optical lattice using the same wavelength. One way of implementing this is by superimposing four separate beams at the position of the atoms and specializing the beams such that each beam pair generates an independent lattice. Another alternative is by using a retroreflected beam to generate the static lattice and two additional beams to produce the moving lattice. This method is difficult to be implemented in our setup. The method that we adopted for our experiment is by combining lattice beams on the same fibers that transport them to the experiment. Adapting this method requires no additional alignment since the lattice beams follow the exact beam path. Each fiber carries
one beam for the static lattice and another beam for the moving lattice. However, these beams have almost the same wavelength but a slightly different frequency. Since the beams have different frequencies and the same polarization in the fibers, they form a beat note. If the frequency of the beat note is comparable to the time scale of the BEC dynamics, it interferes with the experiment. Also, a relatively slow beat note can be sensed by the intensity controller servos and may cause unwanted intensity adjustments. An electronic frequency combination in the AOMs can be used for combining the lattice beams in the fibers. However, the acceptance frequency range of AOMs is limited and large enough frequency detuning between the lattice may not be possible.

To get around these problems, we optically combine the two lattices in the fibers. This method allows for a large frequency difference between the lattice beams and eliminates the possibility of destructive beat notes. In our setup, we utilize the positive order of a 100 MHz AOM to realize the static lattice beams (See Fig. 27). The moving lattice beams are created using the negative order of two 80 MHz AOMs on the right and left of Fig. 27.

Figure 27: Schematics of the moving lattice laser setup. The red beam represents the beam generating the static lattice.

Since the positive order of the a 100 MHz AOM is used (the negative order of the 80 MHz AOM order is used for the moving lattice beams), there is a $\approx 180$ MHz frequency difference between the static and the moving lattice, which would not have a measurable effect on the condensate nor the servos.
However, combining the lattice beams inside the fiber has its disadvantages. Unlike the intensity of the moving lattice beams, the intensity of the static lattice can not be actively and independently controlled after being transported to the experiment. Although, it is still possible to actively control this intensity before it is transported. Stabilizing the laser intensity before the fiber is stable enough for many hours of experimenting.

The feedback signal is picked off from one of the static lattice beams. The photodiode that is shown in Fig. 27 gives the feedback signal to the intensity controller feedback device controlling the 100 MHz AOM. During the experiments, we regularly checked the power balance between the static lattice beams to make sure the laser power is split evenly between them.

### 2.6 TEMPERATURE STABILIZING THE DIPOLE BEAM INTENSITY CONTROLLER

As mentioned in the previous sections almost all of our laser beam intensities are actively stabilized to ensure the experiments are reproducible and heating and excitations are minimized in the condensate. During some of our measurements, we notice a small but measurable atom number drift correlated with the room temperature. The drift in the atom number is especially noticeable when a very cold cloud with smaller atom number is used in the experiment. Since the atom number is sensitive to the dipole evaporation end point, small temperature fluctuations can have measurable effects on the experiments.

We use a simple feedback loop to compensate for the temperature fluctuations. The electronic PCB design is done using Eagle PCB Design software, which can be found in Appendix A. Our goal is to develop a general-purpose, simple, and cost-effective device that can be made quickly.
Figure 28: On the left, the assembled temperature controller box can be seen. The inset shows the finished circuit board PCB prepared in Eagle. On the right, the operating fully assembled temperature controller is presented. The resistor bank is visible on the top right corner. The resistors generate heat to elevate the internal temperature of the intensity controller box above the room temperature. To the left of the resistor bank, the thermistor (red) can be observed.

The results show a significant improvement in the temperature drift of the laser intensity controller after stabilizing the temperature. Figure 29 (a) shows the overnight internal temperature of the intensity controller box while the temperature controller is shut down. Figure 29 (b) shows the temperature overnight after the temperature controller is switched on.

We observe $0.4^\circ C$ temperature drift when we do not actively control the temperature versus $0.1^\circ C$ drift when the temperature controller is engaged. After using the temperature controller, we did not observe significant atom number fluctuations in our experiments.

Figure 29: Overnight temperature of the intensity controller boxes (a) without and (b) with the temperature controller.
2.7 DIRECT DIGITAL SYNTHESIZERS FOR LATTICES AND RAMAN DRESSING

Low noise RF generators are crucial in atomic physics and spectroscopy. Recent developments in digital and integrated electronics have made integrated circuit digital synthesizers possible for very low cost. For example, AD99XX and AD98XX series from Analog Devices are available for under $50 dollars. These chips include digital numerical generators, Digital to Analog Converters, and the clock reference multipliers in the integrated circuitry. However, they require external components to operate. For example, the chip in this experiment requires a precise clock reference for triggering the signal generation, RF filters to reject unwanted harmonics, and a microcontroller to control the input parameters.

Single channel assembled units ”100MHz AD9854 DDS Function Signal Generator Module” are commercially available on eBay for a low price. However, these units come with their own internal clock and therefore one needs to disconnect them and use a synchronized stable oscillator for driving coherent transitions in lattices and Raman coupling.

Figure 30 (a) shows the schematics of a completed device. A GPS (Global Positioning System) receiver generates a 10 MHz reference signal, which is used for driving two units. Each device requires 1 \( V_{\text{rms}} \) sinusoidal signal to operate (to be clocked). Each unit needs an internal clock frequency of at least 2.5\( x \) MHz to generate a stable frequency of \( x \) MHz. To achieve this, an external frequency doubler is used. Together with the internal digital frequency multiplier offered by Analog Devices AD9854, a recommended 300 MHz internal clock is generated.
2.8 DEVELOPMENT OF A SIMPLE MATLAB TOOL FOR SIMULATING ARBITRARY BEAM DIFFRACTION

Our need for simulating the intensity of an arbitrary focusing beam encouraged us to develop a Matlab tool for this purpose. There are many commercial and non-commercial tools available to academia and industry, but developing a tool from ab initio optics is educational. This tool was mainly motivated by simulating the beam intensity of the bilayer Raman beam system discussed in Sec. 2.4. Although the Hermite-Gauss beam modes have analytical forms, having a numerical method is much more versatile as imperfections of the beam or the phase plate can be simulated as well as other situations that can not be implemented using the analytical form.

The main components of our model are source(s), lens(s), and a solution space. The source is simulated by an array of linearly polarized spherical radiators with arbitrary phase and amplitude. The source is assumed to be located just behind the focusing lens. However, this is not a critical assumption, and it is chosen only for convenience. We assume that the phase map of the light field is known at the position of the lens which is reasonable since we use collimated Gaussian beams (with flat phase map) in our experiments. The schematics of the model can be seen in Fig. 31. The primed coordinate system contains the source and the lens, and the unprimed coordinates nest...
the solution space $S$. Since we are interested in the beam intensities close to the beam focus, the position of the focus is placed at the origin of the unprimed coordinate system i.e. the center of the solution space.

![Diagram](image)

Figure 31: The schematics of the model is illustrated in the figure. The primed coordinate system is linked to the source and the lens, and the unprimed coordinate system contains the solution space $S$. The angles $\alpha$, $\beta$, and $\gamma$ are the Euler angles. The source plane is represented with a sand texture.

The basic algorithm works based on adding all the wavelets propagating from the source and superimposing them on every voxel of the solution space $S$. These operations are done using Matlab's internal matrix operations to improve performance. It can be shown that the optical path through the lens and to the focal point is constant across the lens [54]. We use this theorem to develop our algorithm. Figure 32 schematically demonstrates the solution method of our model in 2D for convenience. The source is a plane in space with known intensity and phase. There is an arbitrarily small distance $d$ between the source and the lens, and the focal length of the lens is $f$.

The point $h$ is at the focal plane of the lens. The solid black and blue lines backtrack the focusing beams to the lens all the way to the source. The total phase difference between all the possible paths between the source and the focus point defines the light intensity at that point. In
In this case, the relative phase between the paths can simply be calculated knowing the angle $\theta$. This is because the blue lines behind the lens are all parallel to each other and therefore the relative path length difference between them is $x' \sin(\theta)$. However, finding the intensity at the sample point $e$, which is outside the focal plane, is computationally more involved. In this case, if we backtrack the paths that form a focus at $e$ (dashed lines), we find an imaginary object at point $g$. The red lines behind the lens are the light paths that lead to a focus at point $e$. The difference between the red and the blue lines is that the red lines are not parallel to each other anymore. This makes the calculations slightly more expensive as the path lengths have to be calculated individually for each $x'$ and $y'$ position. This is done by finding the position of the imaginary image at point $e$, which is labeled $g$ in Fig. 32. Then for every $(x', y')$ pair, the local $\sin \theta(x', y')$ is found. The Matlab function used for this operation can be found in Appendix B.

Figure 32: Illustration of the phase calculation based on the focus points. Blue lines are parallel, and they are focused at point $h$. Point $e$ is a result of the red beams being focused. Backtracking these real paths leads to the imaginary dashed-dotted paths that lead to point $g$, which is the imaginary image of $e$. 
CHAPTER 3

RAMAN DRESSING AND SPIN-ORBIT COUPLING OF
BOSE-EINSTEIN CONDENSATES

3.1 INTRODUCTION

In this chapter, we focus on Raman dressing and spin-orbit coupling (SOC) in Bose-Einstein condensates. A series of experiments using Raman dressing techniques had been performed in our group previously [55–57]. Raman dressing is a versatile method for engineering dispersions and excitation spectra because of its tunable parameters [58]. For example, the SOC dispersion can be altered to have a negative effective mass region (see Sec. 3.3), or the excitation spectrum can be engineered to have a roton-like feature in it [17, 59, 60]. In addition to the discussed applications, there are many more applications for SOC in BECs such as simulating Hamiltonians used in solid state physics (e.g. see [15]). We show that a single band model is sufficient for describing dynamics that are slow compared to the Rabi frequency of the Raman coupling in a SOC BEC. Therefore, any dispersion relation that can be approximated by the lower energy band of a SOC system can be simulated using this powerful technique.

In our experiments, Raman dressing is realized by focusing two perpendicular beams with 120 $\mu$m beam waists onto the atoms. The beams cross each other at the position of the condensate and typically their wavelengths are set to 789 nm to avoid inducing dipole forces on the atoms. This might not be the case for some of the experiments as a dipole force might be needed for confining the atoms. The schematic of the experimental setup is presented in Fig. 33. The beams that are used in these experiments are the ones that are diffracted by the 80 MHz AOMs shown in Fig. 11.
Figure 33: Schematic of the Raman setup. The red and the green traces represent the Raman beams.

The required frequency difference between the beams is set by the bias magnetic field and a relatively small detuning $\delta$, which varies depending on the experiment. The spin-orbit coupled system can simply be looked at as a two-state system, which is coupled by a two-photon process with Rabi frequency $\Omega$. Each transition causes a momentum kick equal to the two-photon momentum recoil. The simple picture describing the two-photon Raman process is that the atoms absorb a photon from one beam and emit another photon to the other beam in a coherent process. The momentum transferred to the atoms depends on the angle between the Raman beams. For example, if the beams are co-propagating, no momentum could be transferred to the atoms as the photon recoils cancel each other. Maximum momentum transfer happens for counter-propagating beams in which the photons of the beams have opposite momenta.

Quantitatively, the momentum transferred to the atoms in a two-photon process with laser
wavelengths $\lambda$ and an angle $\theta$ between the beams can be written as

$$\Delta p = 2\frac{2\pi \hbar}{\lambda} \sin(\theta/2).$$

(3.1)

Assuming $\theta = 90^\circ$, Eq. 3.1 simplifies to

$$\Delta p = \sqrt{2} \frac{2\pi \hbar}{\lambda} = 2\frac{2\pi \hbar}{\lambda'},$$

(3.2)

in which $\lambda'$ is called the effective wavelength due to the geometry of the beams.

The beams couple the two states $|F, m_F\rangle = |1, -1\rangle$ and $|F, m_F\rangle = |1, 0\rangle$ of the $F = 1$ manifold. The $|F, m_F\rangle = |1, +1\rangle$ state is decoupled from the process due to the quadratic Zeeman shift. This quadratic Zeeman shift can be calculated using the Breit-Rabi formula shown in Eq. (3.3).

$$E_{|J=1/2,m_J,m_{Jz}\rangle} = -\frac{\Delta E_{hf}s}{2I+1} + g_I\mu_B m_F B \pm \frac{\Delta E_{hf}s}{2} \left(1 + \frac{4m_F x}{2I+1 + x^2}\right)^{1/2},$$

(3.3)

in which $B$ is the magnetic field, $\Delta E_{hf}s$ is the hyperfine splitting between $F = 1$ and $F = 2$ hyperfine states with no magnetic field, and $x = \frac{(g_J - g_I)\mu_B B}{\Delta E_{hf}s}$. $g_J$ and $g_I$ are the electronic and the nuclear Lande $g$-factors, $I$ is the nuclear spin, $\mu_B$ is the Bohr magneton, and $m_F$ is the magnetic quantum number [1]. This equation is visualized for $^{87}\text{Rb}$ in Fig 34.

![Breit-Rabi level scheme for $^{87}\text{Rb}$](image)

Figure 34: Breit-Rabi level scheme for $^{87}\text{Rb}$. 
For example, for a 9.45 G bias magnetic field, there is a 14 kHz quadratic Zeeman shift between the $|1,0\rangle$ and $|1,1\rangle$ states, which is enough to decouple the $|1,1\rangle$ state from the Raman process. The "little bias coils" provide the bias magnetic field (see Fig. 19). The schematic of the Raman beams, their polarization, and the magnetic field direction are illustrated in Fig. 35 (a). Figure 35 (b) schematically elucidates the level scheme and the Raman beams.

![Figure 35](image)

Figure 35: (a) Schematic illustration of the geometry of the Raman system in our setup. The red and the green arrows represent the Raman beams. The laser beams have horizontal polarization to enable both $\sigma$ and $\pi$ transitions. (b) Zeeman sublevels of the $F = 1$ manifold are presented. The colors of the arrows refer to the corresponding laser beams in (a). The dashed red line represents the coupling to the $|1,+1\rangle$ state which is out of resonance.

The intensity and wavelength of the lasers determine the Rabi frequency of the two-level system, and the momentum kick appears as a spatial phase gradient in the off-diagonal elements of the Hamiltonian. This single particle Hamiltonian can be written as

$$H = \frac{p^2_x}{2m} + \frac{\Delta E}{2} \sigma_z + \hbar \Omega \cos (2k_R x - \Delta \omega t) \sigma_x,$$

(3.4)

in which $\Omega$ is the Rabi frequency, $\hbar k_R = \Delta p/2$, $\hbar k_x$ is the momentum of the atom, and $\Delta E$ is the Zeeman splitting between $|F = 1, m_F = -1\rangle$ and $|F = 1, m_F = 0\rangle$ states. The frequency difference between the Raman beams is defined as $\Delta \omega$. The coupling constant $\hbar \Omega$ is given by

$$\frac{\Omega_1 \Omega_2}{2\Delta},$$

(3.5)

in which $\Omega_{1,2}$ are the single photon Rabi frequencies and $\Delta$ is the single photon detuning from the
higher states.

Using the rotating wave approximation \(2 \cos (\omega t) \approx \exp(i\omega t)\), the Hamiltonian can be written as

\[
H = \frac{p_x^2}{2m} + \frac{\Delta E}{2}\sigma_z + \frac{\hbar \Omega}{2} \cos (2kRx - \Delta \omega t)\sigma_x - \frac{\hbar \Omega}{2} \sin (2kRx - \Delta \omega t)\sigma_y
\]

\[
= \begin{pmatrix}
\frac{\hbar \Omega^2}{2} + e^{i(2kRx - \Delta \omega t)} & \frac{\hbar \Omega}{2} e^{i(2kRx - \Delta \omega t)} \\
\frac{\hbar \Omega}{2} e^{-i(2kRx - \Delta \omega t)} & \frac{\hbar \Omega^2}{2} - \frac{\Delta E}{2}
\end{pmatrix}.
\] (3.6)

After transforming the Hamiltonian using the unitary transformation \(U = \exp\{i (kRx - \Delta \omega t/2) \sigma_z\}\), we have

\[
U = \begin{pmatrix}
e^{i(kRx - \frac{\Delta \omega t}{2})} & 0 \\
0 & e^{-i(kRx - \frac{\Delta \omega t}{2})}
\end{pmatrix}
\] (3.7a)

\[
U^\dagger H U = \begin{pmatrix}
\frac{\hbar \Omega}{2} & \frac{\hbar \Omega}{2} \\
\frac{\hbar \Omega}{2} e^{i(kRx - \frac{\Delta \omega t}{2})} & \frac{\hbar \Omega}{2} e^{-i(kRx - \frac{\Delta \omega t}{2})} - \frac{\Delta E}{2}
\end{pmatrix}
\] (3.7b)

Using the Hadamard Lemma and identity, \([x, p] = i\hbar\), we can simplify Eq. (3.7) (b). Therefore,

\[
e^{\pm ikRx} \frac{p_x^2}{2m} e^{\pm ikRx} = \frac{p_x^2}{2m} - \frac{i k_R}{2m} [x, p_x^2] + \frac{1}{2!} \frac{(ikR)^2}{2m} [x, [x, p_x^2]] + 0 + 0 + ... \] (3.8)

Note that the rest of the nested commutation relations are zero since \([x, [x, p_x^2]]\) and the higher order terms vanish. Hence,

\[
U^\dagger H U = \begin{pmatrix}
\frac{p_x^2}{2m} + \frac{\hbar k_R}{m} p_x + \frac{1}{2!} \frac{(\hbar k_R)^2}{2m} + \frac{\Delta E}{2} & \frac{\hbar \Omega}{2} \\
\frac{\hbar \Omega}{2} & \frac{p_x^2}{2m} - \frac{\hbar k_R}{m} p_x + \frac{1}{2!} \frac{(\hbar k_R)^2}{2m} - \frac{\Delta E}{2}
\end{pmatrix}
\] (3.9)

Assuming that \(|\psi\rangle\) is an eigenstate of \(H\), It can easily be shown that \(U^\dagger |\psi\rangle\) is an eigenstate of Hamiltonian \(\tilde{H}\), if

\[
\tilde{H} = U^\dagger H U - i\hbar U^\dagger \frac{\partial U}{\partial \hbar}.
\] (3.10)
Equation (3.10) gives
\[
\tilde{H} = \left( \begin{array}{cc}
\frac{p_x^2}{2m} + \frac{\hbar k_R}{m} p_x + \frac{1}{2!} \frac{(\hbar k_R)^2}{2m} + \frac{\Delta E - \Delta \omega}{2} & \frac{\hbar \Omega}{2} \\
\frac{\hbar \Omega}{2} & \frac{p_x^2}{2m} - \frac{\hbar k_R}{m} p_x + \frac{1}{2!} \frac{(\hbar k_R)^2}{2m} - \frac{\Delta E - \Delta \omega}{2}
\end{array} \right).
\] (3.11)

Using \( \hbar \delta = \Delta E - \hbar \Delta \omega \) and simplifying Eq. (3.11), the Hamiltonian \( \tilde{H} \) can be written as
\[
\tilde{H} = \left( \begin{array}{cc}
\frac{(p_x + \hbar k_R)^2}{2m} + \frac{\hbar \Omega}{2} & \frac{\hbar}{2} \\
\frac{\hbar}{2} & \frac{(p_x - \hbar k_R)^2}{2m} - \frac{\hbar \delta}{2}
\end{array} \right) - \frac{1}{2!} \frac{(\hbar k_R)^2}{2m} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
\] (3.12)

The last term can be neglected since it is a constant. We define \( E_R = \frac{(\hbar k_R)^2}{2m} \) and use it as an energy unit in our SOC experiments. Finally, we can rewrite the effective Hamiltonian in terms of the Pauli matrices:
\[
\tilde{H} = \frac{(p_x + \hbar k_R \sigma_z)^2}{2m} + \frac{\hbar}{2} \sigma_z + \frac{\hbar \Omega}{2} \sigma_x.
\] (3.13)

Diagonalization of Eq. (3.13) gives,
\[
E_{\pm} = \frac{\hbar^2 k_x^2}{2m} \pm \sqrt{\left( \frac{\hbar^2 k_R k_x}{m} + \frac{\hbar \delta}{2} \right)^2 + \left( \frac{\hbar \Omega}{2} \right)^2}.
\] (3.14)

Equation (3.14) represents the single particle energy spectrum of a SOC BEC. The energy spectrum is visualized in Fig. 36 for \( \hbar \Omega = 2.5 \ E_R \) and \( \delta = 2.5 \) kHz. The color coding represents the spin composition of the energy states as it varies with the quasimomentum \( k_x \).
Figure 36: The energy spectrum of the single particle Hamiltonian shown in Eq. (3.13). The color of the curve represents the spin composition of the states. Spin composition is defined as $(|\psi_\uparrow|^2 - |\psi_\downarrow|^2)/(|\psi_\uparrow|^2 + |\psi_\downarrow|^2)$ in which $\psi_\uparrow$ and $\psi_\downarrow$ are the wave functions of the $|1, -1\rangle$ and $|1, 0\rangle$ atomic states, respectively.

### 3.2 Excitation Spectrum of Spin-Orbit-Coupled Bose Einstein Condensates and Emergence of Roton-Like Features

#### 3.2.1 Introduction

J. F. Allen and A. D. Misener experimentally showed that below 2.2 K the hydrodynamics of liquid helium II can not be described by classical hydrodynamics [61]. London suggested that the remarkable properties of liquid helium, what were called “superfluidity” by Kapitza [62], are in fact due to Bose-Einstein condensation [63]. However, to this day, the clear connection between superfluidity and Bose-Einstein condensation is not well known. Tisza suggested that liquid helium II can be well modeled by two penetrating liquid phases [64]: the superfluid phase in which the atoms occupy the ground state of the system and the viscous phase in which the atoms are in the excited states of the system. Landau interpreted Tisza’s theory differently. He treated liquid
helium II as a background with excitations as perturbations to the system, and he did not refer this to Bose-Einstein condensation [65]. At finite temperature, these excitations scatter from each other and the container which makes the liquid viscous [66]. Landau predicted the two fundamentally different excitations in the system: i. Phonons which are sound waves with linear dispersion and ii. Rotons that are interatomic space size quantized vortices. He later corrected his theory to make it more consistent with the thermodynamics of the experimental observations [67]. He predicted a continuous transition from the phonon branch to the roton minimum. Neutron scattering experiments confirmed Landau’s theory with great success [68].

Rotons do not exist in the collective excitation spectrum of Bose gases. However, previous theoretical studies predict a roton-like feature in the excitation spectrum of spin-orbit coupled BECs for small Raman detunings [69,70]. Here, we report on the experimental measurement of these excitations by performing Bragg spectroscopy. In our experiments, we observe the mode softening when the Raman detuning is decreased starting from a finite value. For $^{87}\text{Rb}$ atoms, with the background scattering lengths, a very weak Raman coupling strength and small detuning can lead to a supersolid phase. This supersolid phase is a superfluid with broken translational symmetry. For spin-orbit coupled BECs this phase has been predicted in ref. ??.

For our experimental parameters, the mode softening stops at a finite gap protecting the ground state against the phase transition. Our theoretical analysis corroborates this empirical observation. Furthermore, we find a symmetry in the data for the mode softening which can be explained by a time-reversal like symmetry in the Gross-Pitaevskii Hamiltonian governing the system. Finally, we report on experiments that study the influence of the roton-like mode on the hydrodynamic properties of the BEC.

The content of this section of my dissertation is based on our publication “Measurement of collective excitations in a spin-orbit-coupled Bose-Einstein condensate”, Phys. Rev. A 90, 063624 (2014). For the work presented in this paper, I led the data acquisition and analysis processes. The theoretical analysis of the BdG excitation spectrum was provided by Prof. Yongping Zhang and Prof. Thomas Busch at OIST. All authors contributed to the preparation of the manuscript.
3.2.2 Theoretical Background

As mentioned in the introduction of the chapter, the Raman dressing scheme is based on coupling two Zeeman states of the $F = 1$ atomic hyperfine state. The two states are treated as pseudo-spins, and therefore the BEC resembles an effective spin-1/2 system with a spin-orbit coupled Hamiltonian

\[ H_{\text{soc}} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} - i \frac{\partial}{\partial x} \sigma_z + \frac{\delta}{2} \sigma_z + \frac{\Omega}{2} \sigma_x. \] (3.15)

Here $E_R$ is used as the unit of energy of the system. The equation has become dimensionless by factoring out $\hbar^2 k_R^2 / m$ and replacing $x$ by $k_R x$. Note that $|\Psi|^2$ is also dimensionless by transforming $|\Psi|^2$ to $|\Psi|^2 / k_R$. The spin-orbit coupling is realized by the term $\partial / \partial x \sigma_z$ in Eq. 3.15. The Raman detuning is denoted by $\delta$ and the Raman coupling strength by $\Omega$.

![BdG spectrum and single-particle dispersion](image)

Figure 37: (a) BdG spectrum (solid line) and single-particle dispersion (dotted line) of a spin-orbit coupled BEC for a nonlinear coefficient $g = 0.186$, Raman detuning $0.28 E_R$ ($\delta = 2\pi \times 500$ Hz) and Raman coupling strength $2.5 E_R$. (b) Mode softening with decreasing Raman detuning. The lines correspond to a Raman detuning $1 E_R$, $0.5 E_R$ and 0 from top to bottom.

For the experiments described below, the Raman coupling strength is adiabatically increased starting with a large detuning and ending with a final smaller detuning. This way, the BEC is adiabatically loaded near the global minimum of the lowest spin-orbit band. The quantum state of the BEC is described as the ground state solution of the time-independent Gross-Pitaevskii (GP)
\[ H_{\text{soc}} \Psi + g (|\Psi_1|^2 + |\Psi_2|^2) \Psi = \mu \Psi, \tag{3.16} \]

where \( \Psi = (\Psi_1, \Psi_2)^T \) is the spinor describing the two-components, \( \mu \) is the chemical potential, and 
\[ g = 4\pi a k_R \text{ (since } |\Psi|^2 \rightarrow \frac{|\Psi|^2}{k_R} \text{) with } a \text{ being the s-wave scattering length.} \]
Note that the wavefunction is normalized to the particle number \( N \). We assume that all scattering lengths are equal, which is an excellent approximation for our experimental system [74,75]. While the theoretical analysis neglects the trapping potential present in the experiment, we will show below how this can be compensated for. It is worth noting that the GP equations and the Hamiltonian \( H_{\text{soc}} \) possess a time-reversal-like symmetry described by 
\[ R_\delta K \sigma_x, \]
where \( R_\delta \) flips the sign of the detuning, 
\[ R_\delta R_\delta^\dagger = -\delta, \]
and \( K \) is the operator for complex conjugation. This guarantees the symmetry of ground states of the GP equations when the sign of the detuning is changed. It is also interesting to notice that a similar ground state of the spin-orbit coupled GP equations has been employed in Ref. [76] to produce higher-order partial waves in atomic collisions.

Bragg spectroscopy is a powerful tool to investigate collective excitations in a BEC [77,78]. The linear response to a sudden perturbation through the Bragg pulse can be theoretically analyzed by the Bogoliubov-de Gennes (BdG) equations. The general wavefunction spinor including the ground state, \( \Phi_{1,2} \), and the perturbations, \( \delta \Phi_{1,2} \), can be written as 
\[ \Psi_{1,2}(x,t) = e^{-i\mu t + ikx} [\Phi_{1,2}(x) + \delta \Phi_{1,2}(x,t)], \]
where \( \mu \) and \( k \) are the chemical potential and quasi-momentum of the ground state, respectively. The perturbations can be parameterised as 
\[ \delta \Phi_{1,2}(x,t) = U_{1,2}(x) \exp(iq x - i\omega t) + V_{1,2}^*(x) \exp(-iq x + i\omega^* t), \]
where \( U, V, q, \) and \( \omega \) are the two amplitudes, the quasi-momentum and the frequency of the perturbations, respectively. After substituting the general wavefunctions into the time-dependent GP equations and considering the linear terms in the perturbations, we obtain the BdG equations

\[
\begin{pmatrix}
H_1(k) + (q + k) & \frac{\Omega}{2} + g \Phi_1^* \Phi_2^* & g \Phi_1^2 & g \Phi_1 \Phi_2 \\
\frac{\Omega}{2} + g \Phi_1^* \Phi_2 & H_2(-k) - (q + k) & g \Phi_1^2 & g \Phi_2 \\
-g \Phi_1^* \Phi_2^* & -g \Phi_1^* \Phi_2^* & -H_1(-k) + (q - k) & -(\frac{\Omega}{2} + g \Phi_1^* \Phi_2) \\
-g \Phi_1^* \Phi_2 & -g \Phi_2^* & -(\frac{\Omega}{2} + g \Phi_1^* \Phi_2) & -H_2(-k) - (q - k)
\end{pmatrix}
\begin{pmatrix}
U_1 \\
U_2 \\
V_1 \\
V_2
\end{pmatrix}
= \omega
\begin{pmatrix}
U_1 \\
U_2 \\
V_1 \\
V_2
\end{pmatrix},
\tag{3.17}
\]
where $H_1(k) = -\mu + \frac{(q+k)^2}{2} + 2g|\Phi_1|^2 + g|\Phi_2|^2 + \frac{\delta}{2}$ and $H_2(k) = -\mu + \frac{(q+k)^2}{2} + g|\Phi_1|^2 + 2g|\Phi_2|^2 - \frac{\delta}{2}$.

Figure 38: Roton-like minimum softening and energetic instability for decreasing Raman detuning $\delta$ in a miscible regime ($g_{11} = g_{22} = 0.186, g_{12} = 0.08$) with $\Omega = 2.5E_R$. The lines correspond to the Raman detuning of $0.6E_R$, $0.4E_R$, $0.2E_R$ and 0 from top to bottom.

The resulting BdG spectrum and, for comparison, the single-particle dispersion of $H_{soc}$ for realistic parameters are shown in Fig. 37(a). The difference between the BdG and single-particle spectrum is shown. The BdG spectrum is characterized by a phonon-maxon-roton-like feature with a local minimum near $1.62\hbar k_R$, which is reminiscent of a roton minimum. Reducing the Raman detuning allows to soften the roton-like mode, i.e. to decrease the excitation energy at the position of the minimum, without significantly affecting the long-wavelength phonon modes (Fig. 37(b)). However, for practical parameters using $^{87}$Rb, the energy of the minimum possesses a finite value even when the detuning vanishes. This is in contrast to the behavior of the single-particle spectrum, where the corresponding minimum goes to zero so that the two minima of the double well become degenerate. The finite gap in the BdG spectrum stabilizes the ground state and prevents a phase transition.

We have checked numerically that in the phase miscible regime of scattering lengths, which allows a supersolid as a ground state [79,80], the softening of the roton-like minimum leads to a closure of the gap when a critical value of the Raman detuning is reached. The results are shown in Fig. 38 for a case of miscible parameters $g_{11} = g_{22} = 0.186, g_{12} = 0.08$. Below the critical value
the roton-like modes possess negative energy, which indicates that a state occupying one of the two double-well minima is energetically unstable. The supersolid phase, which is a superposition of components at both minima, is then energetically preferred.

### 3.2.3 Experimental implementation

To experimentally observe mode softening we perform Bragg spectroscopy on a $^{87}$Rb BEC. Our experiment starts with a BEC confined in a crossed dipole trap with harmonic trap frequencies given by $(\omega_x, \omega_y, \omega_z) = 2\pi \times (39, 189, 153) \text{ Hz}$. Spin-orbit coupling in the $x$-direction is induced by two Raman laser beams with $\lambda_R \approx 789 \text{ nm}$, which intersect at the position of the BEC and are arranged with an angle of $90^\circ$ between each other.

The Raman lasers couple the $|1, 0\rangle = |\downarrow\rangle$ and $|1, -1\rangle = |\uparrow\rangle$ hyperfine states in the $F = 1$ manifold, which are split by a 10 G magnetic bias field. The accompanying quadratic Zeeman shift places the $|1, +1\rangle$ state $7.8 \, E_R$ away from resonance.

**Bragg Spectroscopy**

For the Bragg spectroscopy, two laser beams with wavelength $\lambda \approx 1540 \text{ nm}$ and small frequency difference, $\Delta \nu_{\text{Bragg}}$, are pulsed on for 1 ms. The Bragg beams follow the same path as the Raman beams. The Bragg pulse intensity is chosen such that after the pulse $\approx 30\%$ of the atoms are transferred to the $2\hbar k_{\text{Bragg}}$ momentum state shown in Fig. 39 (a). Since the beams are collinear with the Raman beams (see Fig. 33, 1540 nm lasers), $k_{\text{Bragg}}$ is equal to $2\pi/(\sqrt{2} \times 1540 \text{ nm})$. 
The BdG analysis presented in Fig. 38 describes a homogeneous BEC, whereas the experimental system is confined in a harmonic trap. To solve this discrepancy and account for the spatial variation of the density in the experiment, one can introduce an effective scattering length, $a_{\text{eff}}$. To determine the value of $a_{\text{eff}}$ for our experiment, we first measure the Bragg spectrum for a BEC with $10^5$ atoms without spin-orbit coupling (red dots in Fig. 39(b)) and find a peak located at $\Delta \nu_{\text{Bragg}} = 2.7$ kHz. This peak position can be reproduced by the formula for the BdG spectrum of a homogeneous BEC if the density is taken to be equal to the central density of the trapped BEC and all scattering lengths are set to $a_{\text{eff}} = 53.7a_0$ where $a_0$ is the Bohr radius [74]. Due to the weakness of the trap in our experiment, it is a reasonable assumption that the same value of $a_{\text{eff}}$ is valid for the case of a BEC with spin-orbit coupling and we will show below that this leads to an excellent agreement between theory and experiment. To demonstrate the dependence of the BdG spectrum on the interatomic interactions, we also show in Fig. 39(b) the spectrum for a BEC without the spin-orbit coupling that has been released from the trap and allowed to expand for 3 ms before the application of the Bragg pulse (blue squares). In this case, the interaction energy has predominantly been transformed into kinetic energy and the measured peak position, 1998 (17) Hz, is close to
value expected for a single particle, \( E/h = (2\hbar k_{\text{Bragg}})^2 / 2m \hbar = 1934 \text{ Hz} \). This corresponds to the kinetic energy of an atom after receiving a momentum kick of \( 2\hbar k_{\text{Bragg}} \) during the Bragg pulse.

Figure 40: (a) Bragg spectrum for a spin-orbit coupled BEC, measured for \( \hbar \Omega = 3.5E_R \) and \( \delta = 2\pi \times 500 \text{ Hz} \). Each point is an average over four measurements. (b) Schematic of the transitions corresponding to the three peaks in the spectrum. (c) Experimental ToF image of the condensate. The atoms diffracted to the left are due to the Bragg pulse. The number of diffracted atoms enclosed by the dashed ellipse is used for measuring the peak’s \( \alpha, \beta, \gamma \) heights.

Next, we apply Bragg spectroscopy to a BEC with spin-orbit coupling. To observe the most interesting roton-like feature and its softening, the Bragg beam geometry, wavelength, and the spin-orbit coupling parameters should be chosen such that the momentum kick imparted by the Bragg pulse, when directed towards the roton, can transfer atoms to the roton minimum. For our geometry and Raman laser wavelength, this leads to the choice of \( \hbar \Omega = 3.5E_R \) for the data in Fig. 40 and Fig. 41. The resulting roton feature is slightly less pronounced than the one shown in Fig. 37. Spin- and momentum resolved time-of-flight imaging subsequently allows to detect the number of atoms scattered by the Bragg pulse. We find that each Bragg spectrum contains several distinct peaks. A typical spectrum for a BEC with \( 4 \times 10^4 \) atoms, a Raman detuning of 500 Hz and a Raman coupling strength of \( 3.5E_R \) is shown in Fig. 40(a). Each peak corresponds to a different Bragg resonance within the BdG band as indicated in Fig. 40(b). The peak \( \beta \) probes the region of the roton-like mode. The reduced amplitude of the peaks with positive \( \Delta \nu_{\text{Bragg}} \) is due to the
difference in spin composition of the initial and final states of the Bragg transition: unlike the Raman beams, the Bragg beams do not change the spin state due to their large detuning from the Rb D1 and D2 lines.

![Graph](image)

Figure 41: Mode softening. Position of Bragg peaks as a function of Raman detuning. Each point is an average over four data runs. This data was taken for $\hbar\Omega = 3.5E_R$. Vertical error bars are on the order of the symbol size. The data quality in the uppermost branch is impacted by the smallness of the spin overlap between the initial and final state.

To demonstrate the mode softening, we measure Bragg spectra for a range of different Raman detunings, which effectively determine the relative importance of the mean-field interaction vs. spin-orbit effects, and record the positions of the Bragg peaks, see Fig. 41. The three data sets shown correspond to the peaks $\alpha$ (lowest lying curve), $\beta$ (middle curve) and $\gamma$ (highest lying curve) of Fig. 40. One can see that the peak positions significantly shift as a function of the Raman detuning, with the roton mode ($\beta$ peak, middle curve) clearly softening for a decreasing positive value. When the Raman detuning becomes negative, the shape of the dispersion relation of the SOC BEC changes in such a way that the ground state of the BEC now has a quasimomentum opposite to the value at positive detuning. The symmetry between the data points for positive and negative detuning provides direct evidence for the existence of the time-reversal-like symmetry $R_4K\sigma_x$. The fact that the energy of peak $\beta$ never reaches zero corresponds to the absence of a supersolid phase transition.

To demonstrate the excellent agreement between the data and the model using the homogeneous
BdG equations with the effective interatomic scattering length of $a_{\text{eff}} = 53.7a_0$ determined in the context of Fig. 39, we overlay the data in Fig. 41 with the theoretically obtained curves. The effective scattering length leads to a nonlinear coefficient in the GP Eq. (3.16) of $g = 0.186$. The calculated spectrum is in very good agreement with the experimental data.

Figure 42: (a) Spin composition of a spin-orbit coupled BEC after a repulsive light sheet has been swept through it. The shaded rectangle indicates the region where the roton minimum starts to disappear. The roton does not exist to the right of this region. The data was taken for $\hbar \Omega = 2.5E_R$, leading to very obvious roton-like structures for low Raman detunings as shown in Fig. 37. (b) Direction of the motion of light sheet for the two data sets. Drawing not to scale.

The existence of modes carrying roton-like structures can have direct consequences for the hydrodynamic behavior of the BEC. As a particular example we probe the response of a SOC BEC to a repulsive light sheet that is swept through the condensate along the x-direction, see Fig. 42. The light sheet is formed by a laser with a wavelength of 660 nm and Gaussian waist sizes of $w_x = 12\mu m$ and $w_y = 70\mu m$. The central barrier height is approximately three times larger than the chemical potential of the BEC and the sweep velocity of 2.5 mm/s exceeds the central speed of sound ($\approx 1 \text{ mm/s}$). Note that a barrier sweep with these parameters leads to significant heating of the BEC. After the sweep we measure the spin composition $\frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}}$ of the cloud, where $N_{\uparrow}$ ($N_{\downarrow}$) is the number of atoms in the $|\uparrow\rangle$ ($|\downarrow\rangle$) state in the low-momentum components after time-of-flight imaging. Two different scenarios are shown in Fig. 42(b): a barrier moving towards the quasimomentum of the roton (red filled circles), and a barrier moving in the opposite direction (black filled squares). When the Raman detuning is chosen such that the BdG spectrum supports
a roton minimum (i.e. to the left of the shaded region in Fig. 42(a)), a significant difference in spin composition for the two cases is clearly visible. For spin-orbit parameters that do not support a roton minimum (i.e. to the right of the shaded region), the difference in spin composition is much reduced. The fact that this change occurs around the region where the roton minimum disappears possibly indicates the excitation of the roton mode by the moving light sheet. The fact that the existence of the roton minimum plays a role in the spin composition of the condensate suggests that the sweeping light sheet can excite the roton mode. This experiment is reminiscent of exciting the roton mode in superfluid liquid helium. However it is different from liquid helium since the SOC BEC is highly anisotropic because sweeping the barrier in one direction excites the roton mode and the other does not. The exact mechanism of the roton mode excitation by the sweeping barrier is unknown and more detailed investigation of these effects is the topic of future work.

The presented experimental results and their theoretical interpretation will stimulate further studies of the fluid dynamics in SOC BECs. In superfluid helium, the roton minimum limits the flow speed in the superfluid phase according to the Landau criterion. In our system, the Raman dressing, which is employed to generate the spin-orbit coupling, breaks Galilean invariance, and it would be interesting to test the Landau criterion in different moving frames. Furthermore, our experiment can potentially be extended to the case of a spin-orbit coupled lattice [57] where rotons also exist but their manifestation is complicated [81].

3.3 1D EXPANSION OF SPIN-ORBIT COUPLED BOSE-EINSTEIN CONDENSATE

3.3.1 Introduction

In the experiments described in this section, spin-orbit coupled Bose-Einstein condensates are released to expand in a one-dimensional trap (the dipole beam). The asymmetric nature of the expansion motivates us to study the rich physics introduced by the SOC engineered dispersion relations. The opened bandgap between the parabolas associated with the pseudo-spin components
provides inflection points and a negative mass region. Such higher order dispersions, which include more than just a quadratic momentum term, play an increasingly important role in quantum hydrodynamics, fluid dynamics, and optics [82–90]. A negative curvature in the dispersion is interpreted as a negative effective mass as \( m^* = (d^2E/dq^2)^{-1} \). Contrary to common sense, in this situation, a positive applied force will decelerate an object (a wavepacket) moving in the positive direction. In the optical lattice experiments with dilute gas Bose-Einstein condensates (BECs), a negative mass region has been observed to give rise to interesting dynamics [91–95] including the observation of self-trapping [92]. This observation has been the subject of a controversy with respect to the origin of self-trapping. Observing these phenomena such as negative mass hydrodynamics, nonlinearity, breaking of the Galilean invariance and parity in these experiments are great motivation for studying the expansion of a condensate along the SOC direction where the dispersion is modified. This study also motivates simulating the behavior of electrons in lattice bands in solid state physics.

In this section, we experimentally probe the behavior of the BEC in the negative effective mass region of the dispersion, and the results are compared with GP simulations. The BEC is initially prepared and confined in the crossed dipole trap and then is allowed to expand in one dimension in the presence of the SOC [See schematic, Fig. 43 (a)]. Once a sufficiently large phase velocity is achieved, atoms moving on one side of the condensate enter the negative effective mass regime, leading to the slowing of the atoms. The asymptotic velocity of the condensate inside the negative effective mass regime seems to depend on the initial conditions of the condensate as well as the SOC parameters. The negative mass regime also leads to the onset of a modulational instability and causes the formation of soliton trains radiating out the energy.

Figure 43 (a) shows an example of the lower band of the SOC dispersion relation with its associated effective mass. The interaction energy of the condensate causes the BEC to expand and progressively occupy more quasimomentum states of the dispersion.
Figure 43: (a) Schematic illustration of the 1D expansion of the spin-orbit coupled BEC. The asymmetry of the dispersion relation causes an asymmetric expansion of the condensate due to the variation of the effective mass. The effective mass is indicated by the dashed curve and the shaded area represents the negative effective mass region. The parameters used for calculating the dispersion are $\Omega = 2.5 \, E_R$ and $\delta = 1.36 \, E_R$. The color gradient in the dispersion shows the spin composition of the state. (b) Sample ToF images of the 1D expanding spin-orbit coupled BEC for 0, 10, 14 ms are presented. The experimental parameters for the images are the same as for the dispersion in (a).

This section of the dissertation is based on our recent manuscript, which is currently under review. In this work, I led the data acquisition efforts and performed 1D GP simulations using the GPELab Matlab package [96, 97]. These results are presented in this section of my dissertation. My GP simulation results matched well with Prof. Michael Forbes’s 1D and 3D simulation results. The theoretical framework shown in Sec. 3.3.4 is given by Prof. Michael Forbes and Khalid Hossain at WSU. They have also provided careful simulation results for the submitted manuscript. Prof. Yongping Zhang and Prof. Thomas Busch at OIST provided further theoretical input. All authors contributed in preparing the manuscript.

3.3.2 Methods and results

A BEC of approximately $10^5 \, ^{87}Rb$ atoms is prepared in the optical crossed dipole trap ($\lambda = 1064 \, nm$) with trapping frequencies $(\omega_x, \omega_y, \omega_z) = 2\pi \times (26, 170, 154) \, Hz$. The spin-orbit coupling is provided using the Raman laser beams, tuned to approximately the "magic wavelength" ($\lambda_{Ram} = 789.1 \, nm$), coupling $|1, -1\rangle = |\uparrow\rangle$ and $|1, 0\rangle = |\downarrow\rangle$ states in the $F = 1$ manifold which are split by a $\approx 10 \, G$ magnetic field. The quadratic Zeeman shift ($7.8 \, E_R$) decouples $|1, +1\rangle$ from the resonance. The energy unit $E_R$ is defined as $(\hbar k_R)^2/(2m)$, which is approximately $2\pi \hbar \times 1843 \, Hz$ for our setup. Note that $E_R$ is $\frac{1}{4}$ of the recoil energy due to the two photon Raman process and $k_R$ is half of...
the momentum transfer in the two photon Raman process, which is defined as $2\pi\hbar/(\sqrt{2}\lambda R)$. The schematic of the set up is presented in Fig. 44 (a), and the Raman scheme is presented in Fig. 44 (b).

Figure 44: (a) The experimental arrangement of the trap and the Raman beams is illustrated. The angle between the Raman beams is 90°. (b) The Raman coupling scheme in the $F = 1$ manifold is displayed. The $|1, +1\rangle$ state is out of resonance due to the quadratic Zeeman shift.

First, a BEC is formed in the crossed dipole trap and is adiabatically loaded into the SOC states. Then the BEC is released to expand along the SOC direction by turning the vertical beam off. An asymmetric 1D expansion is observed. The experimental results and the GP simulation results are presented in Fig. 45 with excellent agreement. The edges of the BEC are found by fitting straight lines tangent to the end points of the 1D integrated cross section of the condensates. The absorption images are taken in trap. More information about the data analysis process can be found in Sec. 3.3.3. The gray shaded shapes in Fig. 45 schematically represent the absorption images of the expanding BEC. The asymmetric expansion can be observed in the plot shown in Fig. 45.
To demonstrate the impact of the negative effective mass region of the dispersion on the expansion, we increase the Raman coupling strength to $4.5 \, E_R$, which eliminates the negative mass region in the dispersion relation. We observe that the BEC expands symmetrically along the dipole beam. The results are presented in Fig. 46 in which we present the position of the BEC edge versus the expansion time. As we know, beyond $\Omega = 4 \, E_R$, a second order phase transition occurs, and the double well dispersion transforms into a single well. For comparison, we let the BEC expand in the dipole beam without the SOC which exhibits a symmetric behavior with the results shown in the same figure (see Fig. 46). The solid lines are the solution to Eq. (3.18) in which $\lambda_i(t) = \frac{R_i(t)}{R_i(0)}$ with $i = \{x, y, z\}$ and $R_i$ being the Thomas-Fermi radius of the condensate. Equation (3.18) can be derived from the time-dependent generalization of the Thomas-Fermi approximation [98].
have,

\[
\ddot{\lambda}_x = \frac{\omega_x^2}{\lambda_x^2 \lambda_y \lambda_z} \\
\ddot{\lambda}_y = \frac{\omega_y^2}{\lambda_x \lambda_y^2 \lambda_z} - \omega_y^2 \lambda_y \\
\ddot{\lambda}_z = \frac{\omega_z^2}{\lambda_x \lambda_y \lambda_z^2} - \omega_z^2 \lambda_z.
\]

Equation (3.18a) does not have the \(\omega_x^2 \lambda_x\) term because Eq. (3.18) relates to the cloud expansion when the axial confinement is relaxed (by turning the vertical beam off) and the condensate starts expanding in that direction.

Figure 46: Symmetric expansion of a BEC along the axial direction (x axis) with no Raman coupling and strong coupling (4.5 \(E_R\)) is presented. The solid lines are the solution to Eq. (3.18). Beyond \(\Omega = 4 \ E_R\) the double well structure as well as the negative effective mass region disappear in the dispersion relation. The deviation from the theoretical prediction is due to the anharmonic nature of the dipole beam at larger \(x\) coordinates.

Figure 46 shows very good agreement between theory and experiment. Remarkably, no fitting parameters are used in this figure. The only input into the theory are the scattering length, the trapping frequencies, and the TF radius at \(t = 0\). The deviation of the experimental results from the GP and the theoretical curve is due to the anharmonic potential of the dipole beam and the Raman beam dipole force effects.

The dependence of the expansion on detuning, \(\delta\), is studied and the results are presented in Fig. 47. The in-trap expansion is performed using three different detunings namely 0.54 \(E_R\), 1.36 \(E_R\),
and 2.71 $E_R$.

![Graph showing expansion time and edge position](image)

Figure 47: Detuning dependence of the expansion. The curves represent the 1D GP simulation results, and the symbols represent the experimental results. No averaging is done in the experimental results. We observe a meaningful detuning dependence in the right edge (positive coordinate) of the BEC. Larger detuning causes larger expansion rates especially on the right edge of the BEC.

The three-dimensional simulations are also in excellent agreement with the experimental results. The 3D simulations are cylindrically symmetric with $\omega_\perp = 2\pi \times 162$ Hz and the 3D scattering lengths are used in the simulations. In our 1D simulations, the scattering lengths are set such that the TF radius of the ground state matches that of the experiment. The three columns in Fig. 48 show the results for $\delta = 2.71 E_R$, 1.36 $E_R$, and 0.54 $E_R$ from left to right, respectively. The small differences between the experiments and the 3D simulations are mostly due to the thermal effects. On one hand, eliminating thermal effects in the experiments is impossible and reducing it affects the atom number. On the other hand, simulating thermal effects in the condensate is not straightforward, to say the least. Cylindrical symmetry exerts some constraints on the simulations eliminating the purely 3D effects such as turbulence, but the good agreement between the simulations and the experiments proves that they are negligible. A more detailed comparison between the 1D and the 3D simulations will be discussed in Sec. 3.3.4.

In Fig. 48, the corresponding dispersion relations are presented on the top right of each column. The results for the largest detuning ($\delta = 2.71 E_R$) show the most symmetric expansion among the three. This can be explained by considering the fact that at larger detunings the negative mass region is located at larger quasimomenta (look at the dispersion curves from left to right).
Therefore, the quasimomentum states in the negative mass region remain unoccupied during the expansion of the condensate for large detunings because the initial chemical potential of the cloud is not sufficient for occupying those energy states. However, the situation is different for the lower detunings. The middle and the right columns in Fig. (48) clearly show a pileup. The pileup appears on the right-hand side of the cloud for expansion times larger than 11 ms and 8 ms respectively. The time difference in the appearance of the pileup can be explained by the time difference in the occupation of the negative effective mass states. The lower the detuning is the faster the condensate reaches the inflection point and the negative mass region.
Figure 48: Comparison between the results of the 3D simulation and the experiments. The red dotted curves are the integrated densities, and the solid black curves are the integrated theoretical 3D results. The theoretical results are calculated for $\delta = 2.73 E_R$, $1.37 E_R$, and $0.546 E_R$, respectively and the experimental results, from left to right, are measured for $\delta = 2.71(3) E_R$, $1.36(3) E_R$, and $0.54(3) E_R$, respectively. There is a slight difference between the experimental and the theoretical detunings, however the theoretical values lie well within the experimental errorbars.
3.3.3 Experimental considerations

Magnetic field gradient

The bias magnetic field plays a major role in our SOC experiments. Extra care must be taken to make sure that small magnetic field gradients do not affect the results. The approximately 10 G bias magnetic field is realized using the little bias coils (see Fig. 19) generating a magnetic field along the \( x \) axis. Although the little bias coils are in Helmholtz configuration, a slight magnetic field gradient can remain due to the residual magnetic fields. These gradients can occur due to slight setup configuration changes or temperature drifts. Since each coil is controlled via a different current driver, the magnetic field gradient can be canceled by slightly modifying the current in one of the coils. The procedure includes generating a 50/50 mixture of \( |\uparrow\rangle \) and \( |\downarrow\rangle \) states in an elongated trap (dipole beam). A slight magnetic gradient along the weak trapping axis (\( x \) axis) results in a shift of the trap center. The trap shift can be found by balancing the magnetic gradient force and the trap force. Therefore,

\[
m\omega_x^2 x_0 = \mu_B g_F m_F \frac{\partial B_x}{\partial x},
\]

(3.19)

in which \( x_0 \) is the shift in the trap center due to the magnetic field gradient, \( \mu_B \) is the Bohr magneton, \( g_F \) is the g-factor for the hyperfine state, \( m_F \) is the projected angular momentum, \( m \) is the atomic mass, and \( \omega_x \) is the trapping frequency. For example, a magnetic field gradient of 78.2 mG/m shifts the center of the trap by 128 \( \mu \)m assuming the trapping frequency \( \omega_x \) is 1.4 Hz.

This relative shift can be detected using the Stern-Gerlach ToF imaging procedure (look at Sec. 2.2.9). Figure 49 shows one example of this test. In a successful test, similar to what is shown in Fig. 49, different atomic state clouds approximately rest at the same horizontal position of the elongated trap.
Figure 49: ToF image of a two component ($|\uparrow\rangle$ and $|\downarrow\rangle$) BEC after being held in the dipole trap for 15 s. The species are separated using a vertical Stern-Gerlach field. This image is taken for one of our calibration measurements for the data presented in this section. The reason the contrast is lower than usual is that it is taken using the $1 \rightarrow 1'$ atomic transition.

**Adiabatically loading the BEC into the SOC states**

To ensure the prepared BEC is in the ground state of the SOC dressed states we make sure that the procedures are slow compared to the time scale of the system. For example, dipole or quadrupole excitations occur on the time scale of the trapping period. Therefore the experimental procedures have to be performed on much longer time scales than the trapping period. Also, to make sure that the condensate remains in the lower SOC dressed energy band, the preparation time scale has to be much longer than the SOC Rabi frequency $\Omega$. The procedure of adiabatically loading the BEC into the SOC state includes:

- Enabling the magnetic field stabilizer feedback system 4 s before the end of the dipole evaporation procedure.
- Ramping on the magnetic field to approximately 10 G in 100 ms. The magnetic field is also turned on approximately 4 s before the end of the dipole evaporation sequence.
- Simultaneously ramping on the Raman lasers in 100 ms while they are at least 5 kHz detuned from the final detuning $\delta$.
- Ramping down the detuning in 100 ms from $5 \text{ kHz} + \delta$ to $\delta$.

At this point, the BEC is in the ground state of the SOC dressed state, and the vertical trap is switched off to start the expansion.
Centering the laser beams

Since the 1D expansion of the BEC is studied in these experiments, it is critical to making sure that the potential around the BEC is symmetric around the origin. This is accomplished by centering the laser beams around the center of the optical trap. First, we start with centering the dipole beam by aligning the focus of the beam to the center of the crossed dipole trap to assure a symmetric BEC expansion. Small day to day changes in the optical table tilt can cause a slightly different rest position for the BEC. This can be compensated by adjusting the translation stage that supports the dipole beam focusing achromat lens. In addition to centering the dipole beam, the Raman beams need to be centered around the origin. This is mostly due to the slight dipole force of the Raman lasers on the atoms. Although this effect is minimized by tuning the laser wavelength to the “magic wavelength”, Raman dressing applies some dipole force to the atoms by lowering the energy of the dressed states. Stronger Raman coupling lowers the energy of the dressed states even more. Therefore, the Gaussian profile of the Raman beams causes an attractive force on the atoms towards the center of the Raman beams because the energy is minimized there.

Magic wavelength

The magic wavelength is referred to a laser wavelength that imposes no dipole force in the following sense: If the laser frequency is carefully tuned between two atomic transitions, the total dipole contribution of the atomic transitions cancel each other and leave no total energy shift in the dressed atomic states. Depending on the wavelength of the Raman beams, a small dipole force might cause unwanted trapping potentials that can change the trapping geometry. To minimize this effect, we initially adjust the laser wavelength to approximately 789.1 nm which is close to the magic wavelength. However, as briefly mentioned previously, the spin-orbit coupling itself slightly reduces the ground state energy depending on the Raman coupling strength Ω and the detuning δ. Figure 50 (a) shows the normalized combined Raman beam intensities looked at from the top (looking towards the $-z$ direction). Figure 50 (b) shows a slice of the same data along the $x$ axis for $y = 0$. The intensity of the Raman beams falls below 90% for distances larger than 39.2 μm
from the center. The simulation shows that the energy shift due to the variation of the Raman laser intensities across the BEC remains small in our experiments.

![Figure 50: Raman laser beams intensity simulation. (a) Numerical results of the normalized combined laser intensities of the Raman beams at the position of the atoms looked from the top (looking towards $-z$ direction). (b) Calculated Raman laser intensities along the $x$ axis. The combined Raman intensities fall below 90% for $|x| > 39.2 \mu m$.](image)

Finding the Raman zero detuning

The process of finding the Raman zero detuning starts with a BEC in the crossed dipole trap. Then the Raman beams are ramped on one after the other while they are tuned to the intended detuning $\delta$. Similar to adiabatically loading the BEC into the spin-orbit coupled states, the bias magnetic field, and the magnetic field stabilizer are turned on 4 s before the end of the dipole evaporative cooling. The BEC is kept in the Raman beams with a fixed detuning for approximately 500 ms. Then the laser beams are turned off, and a Stern-Gerlach ToF absorption imaging procedure is performed to map the spin state of the condensate by counting the number of atoms in each state. Zero Raman detuning is achieved when the total spin composition of the atoms is zero. Note that when $\delta = 0$, the SOC dispersion relation is symmetric.

Expansion anomalies for lower detunings

We observe a good agreement between the theory and the experiment for Raman detunings $\delta = 2.71 \ E_R \approx 5 \ kHz$, $1.36 \ E_R \approx 2.5 \ kHz$, and $0.54 \ E_R \approx 1 \ kHz$ as shown in Fig. 48. Although
the expansion of the cloud matches the theory on the right-hand side of the cloud, we realize an anomaly on the left-hand side for smaller detunings. Note that, on the left-hand side of the cloud the dispersion is quadratic and we do not expect to see any pileup (for example look at Fig. 43).

The left panel of the Fig. 51 shows the expansion of the condensate for detunings 0.54 $E_R \approx 1$ kHz, 0.41 $E_R \approx 750$ Hz, 0.27 $E_R \approx 500$ Hz, and 0.14 $E_R \approx 250$ Hz. The anomalies can be observed in both panels indicated by the curly brackets. For example, on the left panel and for smaller positions, we see a density deprived area. The stopping seems to be most severe for $\delta = 500$ Hz.
Figure 51: In-trap expansion of the condensate is shown on the left column for detunings $0.54 E_R \approx 1$ kHz, $0.41 E_R \approx 750$ Hz, $0.27 E_R \approx 500$ Hz, and $0.14 E_R \approx 250$ Hz from top to bottom, respectively. In the right column, a time slice of the 1D cloud density is presented.

We suspect that thermal effects are responsible for observing these anomalies. In this case, the mechanism for the pileup on the left-hand side can be explained as follows: Due to thermal excitation, a slight population of the atoms occupies some states in the second well in the dispersion relation. Similar to the atoms in the ground state, these atoms start expanding as the trap is relaxed. For these atoms, the negative mass region is located on the left-hand side of the dispersion opposite to the atoms in the ground state. This can explain why the anomaly appears on the left-hand side of the condensate as it expands. However, occupying the second well in the dispersion implies that
most of the excited atoms have to have the $|\downarrow\rangle$ atomic state. This can be verified by looking at the Stern-Gerlach ToF images. Figure 52 shows two Stern-Gerlach ToF images with $\Omega = 2.5 \ E_R$ and $\delta = 1 \ \text{kHz}$ at the temperatures that our experiments were normally performed. Figure 52 (a) has 8 ms in-trap expansion and shows some atoms in the $|\downarrow\rangle$. These atoms were not observed in the earlier in-trap expansion times. If our observation is accurate, this means that after several milliseconds of in-trap expansion, some hotter atoms find themselves in the well on the right-hand side of the dispersion. Figure 52 (b) shows the condensate after 9 ms in-trap expansion time.

![Figure 52: Stern-Gerlach ToF absorption images of the condensate after (a) 8 ms and (b) 9 ms in-trap expansion.](image)

**Data processing**

Asymmetric expansion of the cloud is easily recognizable by looking at the experimental results such as in-trap results presented in Fig. 48 or the ToF results in Fig. 43 (b). However, quantifying the observed features in the experiment is not straightforward. As the condensate expands in an asymmetric fashion, fitting a reasonable function to the 1D intensity profile of the condensate becomes a non-trivial problem. Also fitting an empirical function to the entire intensity profile might introduce unreal features to the problem. To minimize or avoid this controversy, we initially started with minimal linear fitting to the edges of the cloud and later used quantiles for more reliable results. In the following, we start with explaining the data preparation process followed by explaining these methods for analyzing the data.

The experimental data shown in Figs. 45, 46, and 47 is generated using the absorption images of the atoms in-trap. No ToF is performed for these images to avoid complications due to the expansion in our simulations. Although ToF images are suitable for observing the details inside the
condensate, they are not suited for quantifying the in-trap expansion rates or the density profiles. Therefore, we use in-trap images for direct comparison with the GP simulation results.

To prepare the raw experimental data for analysis, we start with detrending the 2D data by fitting two individual lines to the top horizontal and the left vertical axes of the experimental images and construct an approximate gradient map of the region of interest (ROI). Then the 2D gradient map is used to detrend the 2D data. The in-trap absorption images are integrated along the $z$ direction inside the ROI to generate a 1D density profile of the condensates (see e.g. [99–101]).

Our linear edge finder algorithm is implemented by first detrending and smoothing the 1D data. Then the data is smoothed using a SavitzkyGolay filter such that it only suppresses the high-frequency noise (mostly camera pixel noise), and leaves the experimental effects unaffected. Then an intensity setpoint is chosen such that it vertically bisects the 1D intensity profile. It must be noted that the same setpoints were used for both the experiment and the simulation results for consistency. Then two lines are fitted to a few points of the smoothed data just smaller than the setpoint. The $x$ intercept of the fitted lines are then used for approximating the edges of the cloud. The results of this method are reasonably reliable, and it catches the essence of the experiment and the simulation. The results shown in figures 45, 46, 47 are found using this method. However, this method has its weaknesses. For example, it is too sensitive to fringes and noise that randomly appear in the experimental images. The method of quantiles explained in the next paragraph attempts to address this weakness. Figure 53 shows an example of the edge finding procedure for an integrated atomic density. The blue curve is the original signal that is obtained by integrating a 2D image to get a 1D signal. The orange-red curve is the detrended 1D signal, and the black curve is the smoothed signal. The orange line represents the setpoint, and the thick green (red) lines are the linear fits on the right (left) to the smoothed curve.
Figure 53: This figure illustrates the method initially used for finding the edges of the cloud. The blue curve is the original 1D signal. The red/orange curve is the detrended data. The black curve is the smoothed data, and the red and green lines are the linear fits to the smoothed curves. The horizontal line on the bottom is the linear fit to the baseline, and the one above is the intensity setpoint, which vertically bisects the data and is used for finding the linear fits (red and green lines).

The method of quantiles relies on calculating the area underneath the 1D intensity curves. For example, a 20% quantile position means the position where 20% of the integrated intensity lies on the left and 80% of the intensity lies on the right-hand side of that position. The strength of this method is that it automatically averages over the high-frequency fluctuations in the intensity and is less sensitive to the local noise. However, this method does not give the position of the edges of the cloud, and for example, one can not find the TF radius of the cloud using this method. It is most accurate for values closer to the 50% of the total density since close to the center of the cloud, the slope of the accumulative sum is maximum. Another disadvantage of using quantiles is the systematic effect of the thermal cloud around the condensate because the thermal cloud is always included in the accumulative sum. Figure 54 represents 20 quantiles covering the cloud. As mentioned above, the quantiles that are closer to the edges of the cloud show more noise since the slope of the accumulative sum is smaller.
3.3.4 Theory

Similar to Eq. (3.13) the dynamics of the system is governed by a set of two coupled of GPEs describing the two spin components. Here, we offset the energy levels by the chemical potential $\mu$ for convenience. We have,

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} = \begin{pmatrix} \frac{p^2}{2m} + V^{\uparrow} & \frac{\Omega}{2} e^{2ikRx} \\ \frac{\Omega}{2} e^{-2ikRx} & \frac{p^2}{2m} + V^{\downarrow} \end{pmatrix} \cdot \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix},$$

$$V^{\uparrow} = -\mu + \frac{\delta}{2} + g_{\uparrow\uparrow} n^{\uparrow} + g_{\uparrow\downarrow} n^{\downarrow},$$

$$V^{\downarrow} = -\mu - \frac{\delta}{2} + g_{\downarrow\uparrow} n^{\uparrow} + g_{\downarrow\downarrow} n^{\downarrow},$$

where $\hat{p} = -i\hbar \hat{\nabla}$ is the momentum operator, $\delta$ is the detuning, $\Omega$ is the SO coupling strength, $k_R$ is the Raman wave-vector, $g_{ab} = 4\pi\hbar^2 a_{ab}/m$ and $a_{ab}$ are the $S$-wave scattering lengths. For the $|\uparrow\rangle = |1, -1\rangle$ and $|\downarrow\rangle = |1, 0\rangle$ hyperfine states of $^{87}\text{Rb}$, the three scattering lengths are almost equal and we take $a_{\uparrow\uparrow} = a_{\uparrow\downarrow} = a_{\downarrow\downarrow} = a_s$. Although the 3D scattering lengths in a 3D simulation give the more accurate picture of the system, an effective 1D model can be used by tuning the coupling constants appropriately as described in [102]. It must be noted that approximating a 3D
system using a 1D model is only possible if the trap is highly elongated [102]. For the experimental parameters, this effective 1D approximation works quite well (see Fig. 55), but we find some radial excitations, so we compare directly with 3D axially symmetric simulations with $\omega_{\perp} = 162\,Hz$.

Figure 55: Comparison of the full two-component 1D GPE model Eq. (3.20) (top) with the 1D single-band model Eq. (3.22) (middle) and the axially symmetric 3D single-band model (bottom) for detuning $\delta = 0.546E_R$.

**Single Band Model and Spin-Quasimomentum Map**

One of the main theoretical results we wish to convey is that in many cases, a single-band effective Hamiltonian can capture the essential dynamics of a BEC with SO couplings. The two-component dynamics can then be reconstructed from this single-band model using a spin-quasimomentum mapping that expresses the local distribution of spins from the local quasimomentum. We have verified by direct comparison that, for the experiment under consideration, the dynamics are accurately described by such a single-band model.
The idea is that for homogeneous systems, the two-component system in Eq. (3.20) can be diagonalized into two bands. The system will remain almost entirely in the lowest band as long as it is gently excited with respect to the separation of the two bands. Note that the separation of the two bands is proportional to the Raman coupling strength $\Omega$. For inhomogeneous densities, this picture is locally valid for slowly varying densities, similar to the Thomas-Fermi approximation. Furthermore, the approximate equality of the coupling constants allows one to define a spin-quasimomentum mapping that relates the two-component spin populations $n_\uparrow$ and $n_\downarrow$ to the quasi-momentum of the single-component state. This greatly simplifies the analysis and allows us to demonstrate that all of the interesting physics observed in the experiment e.g. asymmetric expansion, the pile-up, slowing down, and instabilities are due to the modified dispersion relationship.

A simplified spatial coordinate independent single particle Hamiltonian is derived using Eqs. (3.4) to (3.12) to obtain Eq. (3.13). The spin-quasimomentum map follows from the composition of the lowest eigenstate as a mixture of $|\uparrow\rangle$ and $|\downarrow\rangle$:

$$\frac{n_\downarrow - n_\uparrow}{n_\downarrow + n_\uparrow} = \frac{k - d}{\sqrt{(k - d)^2 + w^2}}, \quad (3.21)$$

where use $k = p/hkR$, $d = \delta/4E_R$, and $w = \Omega/4E_R$ for compactness. This may be used to reconstruct the spin population of the original two-component system given quasimomentum $hk$. It is worth noting that if the couplings are not equal, then both the dispersion relationship Eq. (3.23) and spin-quasimomentum mapping Eq. (3.21) depend on the density $n$, which complicates the analysis.

Thus, as long as the system is gently excited, the population of the upper branch remains negligible, and one can model the SO-coupled BEC with the following single-component model:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \left( E_\pm(k) + gn + V_{\text{ext}}(x) \right) |\psi\rangle \quad (3.22)$$
where the dispersion $E_-(k)$ has the form

$$\frac{E_+(k)}{2E_R} = \frac{k^2 + 1}{2} \pm \sqrt{(k - d)^2 + w^2}, \quad (3.23)$$

which is the simplified version of Eq. (3.14). The dispersion realized in our experiment for different detunings $\delta$ are shown in Fig. 56. To verify the accuracy of the single-band model Eq. (3.22), we compare it with a full two-component simulation (3.20) in Figure. 55.

Figure 56: Effective single-particle dispersion Eq. (3.23) (solid lines, left axis) and spin–quasi-momentum mapping Eq. (3.21) (dotted lines, right axis) for the experimental parameters $\Omega = 2.5E_R$ ($w = 0.63$) and $\delta = \{2.73, 1.37, 0.546\}E_R$ ($d = \{0.683, 0.341, 0.137\}$) respectively, from top to bottom. The two inflection points along with the middle point, which has the least negative effective mass are shown as dots with thin lines demonstrating the group velocities $v = \frac{\delta E_-(k)}{\partial k}$ shown in Fig. 57. The region of negative effective mass between the inflection points is lightly shaded.

**Hydrodynamics**

The hydrodynamics of such a single-component theory with generalized dispersion $E_-(k)$ can be deduced by using a Madelung transformation $\psi = \sqrt{n}e^{i\phi}$, where $n(x,t) = n_\uparrow + n_\downarrow$ is the total density at position $x$ and time $t$, and $\phi(x,t)$ is the phase, which acts as a quasimomentum potential
\[ p = \hbar \frac{\partial}{\partial x} \phi. \]  

The 1D hydrodynamic equations have a familiar form

\[ \frac{\partial}{\partial t} n + \frac{\partial}{\partial x}(nv) = 0, \]  

\[ \frac{\partial v^*}{\partial t} + v^* \frac{\partial}{\partial x} v^* = -M^{-1} \frac{\partial}{\partial x}[V_{\text{eff}} + V_Q], \]  

\[ M^{-1} = E''(k), \quad v^* = E'(k), \]  

where \( v \) is the group velocity, \( j = nv \) is the current density, \( V_{\text{eff}} = V_{\text{ext}}(x,t) + gn(x,t) \) is the effective potential including the external potential and mean-field effects \([103]\), and \( E'(k) \) and \( E''(k) \) are defined as \( \frac{\partial E_{-}(k)}{\partial k} \) and \( \frac{\partial^2 E_{-}(k)}{\partial k^2} \), respectively. What differs from the usual Madelung equations is that third and higher derivatives of the dispersion \( E_{-}(k) \) affect the velocity \( v^* \) and quantum potential \( V_Q(n,p) \). In this case, for an inhomogeneous matter, \( v^* \) also differs from the group velocity \( v \). Likewise, the quantum potential \( V_Q(n,p) \) contains additional corrections to the usual quantum pressure term \( \propto \left( \frac{\partial}{\partial x} \right)^2 \sqrt{n}/\sqrt{n} \). For roughly homogeneous sections of the cloud, however, these corrections are small: \( v \approx v^* \) and the usual hydrodynamic behavior is realized.

**Self-Trapping**

Key to these equations is the inverse effective mass \( M^{-1} \) which derives from the second derivative of the dispersion. For the effective dispersion \( E_{-}(k) \) realized by the lowest band of the SOC system, the effective mass is negative as shown in Fig. 56. The effect is that when a force \( F = -\frac{\partial}{\partial x} V \) is applied to a wavepacket with quasimomenta in this negative effective mass region, it accelerates backward.

The experimental results may thus be explained qualitatively using a Thomas-Fermi–like approximation where we assume each point of the cloud is locally well described by a plane-wave with local quasimomentum \( p \). Initial equilibrium is established between the external trapping force \( -\frac{\partial}{\partial x} V_{\text{ext}} \) and the internal mean-field pressure \( -\frac{\partial}{\partial x}(gn) \), with the quasimomentum \( k = k_0 \) minimizing the kinetic energy. Once the trapping potential is reduced, the cloud starts to expand due to the mean field pressure since the effective mass is positive, and both the group and phase velocities...
(quasimomentum) increase in magnitude. Initially, the expansion appears symmetric, but as the quasimomentum along the positive $x$-axis approaches the region of negative mass, the acceleration of the cloud along the positive axis slows significantly compared with the acceleration along the negative axis. This asymmetry is seen in Fig. 48 and is a demonstration of the explicit breaking of parity and Galilean invariance in SO coupled systems. Once the quasimomentum enters the region of negative mass, the acceleration opposes the force, and the cloud experiences the “self-trapping” effect where the positive mean-field pressure tends to prevent further expansion.

A similar effect has been observed in optical lattices where the effective mass becomes negative near the edge of the Brillouin zone [91,92,104]. However, these “self-trapping” effects have been attributed to several different phenomena. One picture is based on a Josephson effect due to the effective double-well potential of the optical lattice [92,105], which was predicted from a variational framework [106]. Another explanation is that the sharp boundary is a “gap soliton” [107], though this explanation is disputed by [105] on the basis that solitons should remain stable whereas the latter observe self-trapping only for a finite period. Finally, self-trapping has been explained in terms of the Peierls-Nabarro energy barrier [108].

Here the origin of this self-trapping effect is the appearance of a negative mass in the effective dispersion relationship. The single-band model thus provides a simple explanation of these effects. The analysis of self-trapping in optical lattices is complicated by the coexistence of both a modified dispersion arising from the band structure and a spatial potential (Josephson effects). In this regard, the SO-coupled BEC we study here is simpler – the background potential is homogeneous, and the only effect is that of a modified dispersion. A single-band model was used to reproduce the results in [105] with a cosine dispersion relationship, which follows from a tight-binding approximation of the optical lattice in [105]. With this single band model, we reproduce the self-trapping and observe the continued increase in the RMS-width of the cloud after an initial trapping. In our simulations, we find that the boundaries of the trapped region remain motionless for extended periods of time, but high-momentum particles can leak through the boundary. These high-momentum particles are responsible for the continued increase in the cloud radius. Note that in our simulations, the
boundary appears to be very stable, but is “leaky”. As can be seen from Fig. 57, the boundary maintains its shape, but a small number of fast moving particles can escape. These fast-moving particles are responsible for the continued increase in the width of the cloud seen by [105] even though the boundary remains stopped.

Figure 57: Plot of the total density, \( n = n_\uparrow + n_\downarrow \), as a function of time for the single-band 1D GPE simulation of the in-situ expansion described in the text. Top: Full image demonstrating the asymmetric expansion seen in the experiment. Bottom: Zoom into the region where the modulational instability first appears. The dashed lines are the three group velocities \( v_g = E'_\uparrow(k) \) at the quasi-momenta \( k \) where the effective mass \( m_s^{-1} = E''(k) \) starts becoming negative (steepest line), the point of minimum negative effective mass (middle), and where the mass returns to positive (least steep line). The red dots indicate the region with negative effective mass.

The question naturally arises as to what sets the limiting velocity of the expanding edge. The GPE simulations clearly show that (see Fig. 58) this limiting velocity lies fully inside the negative mass region, close to the group velocity associated with the point of most negative inverse mass. This is in contrast with [91] where they claim to observe the limiting velocity at the inflection point where the mass first starts to become negative. While this qualitatively describes the limiting
velocity, the full effect is somewhat subtle. The limiting velocity ultimately depends on several factors, including the preparation of the system [109], but requires a quasi-stable boundary, which is tied to the negative mass through a dynamical instability as we discuss later.
Figure 58: Comparison of in-situ images from experiment (top of each pair) with the 1D GPE simulations (bottom of each pair) for different detunings $\delta = (2.71(3), 1.36(3), 0.54(3)) E_R$, respectively from top to bottom. Upper dashed white lines show the three group velocities $E'_-(k)$ shown in Fig. 56. The lower dashed white curve is the integral of the group velocity $x(t) = x_0 + \int E'_-(k(t)) \, dt$ where $k(t)$ is the quasi-momentum at the boundary as defined in Fig. 59.

Having validated the single-component GPE model Eq. (3.22), we explore some of its behavior.
In Fig. 59 we show the bulk behavior of this model for the current experimental parameters of a qualitative level the behavior is as follows.

The initial cloud at $t = 0$ is symmetric. Galilean invariance is broken by the spin-orbit coupling, but as this only modifies the dispersion relationship, it does not appear in the ground state. The size of cloud along the $x$ axis (tens of microns) is long compared with the healing length (hundreds of nanometers) of the system, and the state can be well-described by the Thomas-Fermi approximation. Essentially, each point in the trap can be thought of as roughly a plane-wave solution with a local quasimomentum $k(x)$. In the ground state $k$ sits at the minimum of the dispersion relationship in equation (3.23).
Dynamical Instability

Another feature evident from the experiment is the emergence of large-amplitude modes near the boundary. This feature can be seen in the ToF experimental images such as Fig. 43 (b) for the 14 ms in-trap expansion case. These low-energy excitations, the phonons, can be studied by performing a BdG analysis of the homogeneous system. A novel feature of the SO coupling is that the qualitative
structure of the phonon spectrum depends sensitively on the quasimomentum $k$ of the background state. A phonon with momentum $q$ on a background of quasimomentum $k$ has frequency:

$$\hbar \omega_q = E_- \pm \sqrt{E_+(E_+ + 2gn)},$$

(3.25)

where

$$E_- = \frac{E_-(q + k) - E_-(q - k)}{2},$$
$$E_+ = \frac{E_-(q + k) + E_-(q - k) - 2E_-(q)}{2}.$$

For small quasimomenta, the excitation spectrum is linear $\omega_q \propto c_{\pm}|q|$ where the speed of sound $c_{\pm}$ is generally different in the positive and negative directions. For larger momenta, the excitation spectrum exhibits a roton-like feature [59,69] reminiscent of the $R^-$ roton seen in liquid Helium [82] and shaken optical lattices [17]. This roton-like feature is also responsible for a finite normal component at $T = 0$ [110].

For this experiment, the crucial feature of the quasiparticle spectrum is that when $E_+ < 0$, $\omega_q$ acquires an imaginary component. The corresponding phonon mode will grow exponentially in amplitude, resulting in a dynamical instability [8,20]. From the form of $E_+$ we see that in the limit $q \to 0$ that $E_+ \propto E''_-(k)$. Thus, for any quasi-momentum $q$ at which the effective mass $m_* < 0$ is negative, there will be an instability. (The instability appears for finite $q$ when the effective mass is still positive, but for the situations under consideration, these regions are indistinguishable from the negative mass regions). From the GPE simulations, the picture emerges that once the cloud enters the negative effective mass region, small fluctuations grow exponentially forming the sharp boundary of the cloud. Initially, these growing modes appear chaotic, but as is typical with dispersive shock waves (see [89] and references therein), energy is “radiated” from the boundary in phonons and soliton trains. These trains are clearly visible in Fig. 59. As energy is dissipated, the boundary appears to stabilize due to non-linear effects. This stabilization appears to be critically connected to the negative mass: A similar boundary with standard dispersion does not stabilize,
but instead broadens [109]. The exact mechanism is not yet clear, but the dynamical instability appears to keep the fluctuations at the boundary large, preventing the boundary from broadening.

### 3.4 MAGNETIZATION OF LATTICE DRIVEN SPIN-ORBIT COUPLED BOSE-EINSTEIN CONDENSATES

#### 3.4.1 Introduction

In SOC systems the quasimomentum of the ground state dictates the spin composition of the condensate. The dispersion itself can be uniquely defined knowing the SOC coupling strength Ω, the detuning δ, and the SOC strength $\frac{\hbar k^2}{m}$ in which $k_R$ is half of the Raman process momentum transfer and $m$ is the atomic mass. A lattice coupling the two minima of the SOC dispersion can have fascinating consequences such as driving a coherent coupling between the two minima leading to a transient formation of the stripe phase. In this section, we provide experimental evidence for such coherent coupling.

The experimental results provided in this section are obtained in our group at WSU, and I led the experiments. Theoretical results presented in Sec. 3.4.3 are performed by our theorist collaborators Dr. Kuei Sun and Prof. Chuanwei Zhang at UT Dallas.

#### 3.4.2 Experimental implementation and results

In this section, we briefly discuss our experimental methods and results. In order to realize spin-orbit coupling with the optical lattice modulation we start with a BEC of $\approx 10^5$ $^{87}$Rb atoms inside the crossed dipole trap with trapping frequencies $(\omega_x, \omega_y, \omega_z) = 2\pi(24, 170, 150)$ Hz. An approximately 10 G bias magnetic field generates a 7 MHz Zeeman shift between the states $|F, m_F\rangle = |1, 0\rangle$ and $|F, m_F\rangle = |1, \pm 1\rangle$. The perpendicular Raman beams with $\lambda_R \approx 789$ nm intersect at the position of the condensate (schematics shown in Fig. 60(a)). The Raman lasers couple the two states $|1, -1\rangle = |\uparrow\rangle$ and $|1, 0\rangle = |\downarrow\rangle$, while $|1, +1\rangle$ is decoupled from the SOC scheme due to the quadratic Zeeman shift. The two photon Raman process is detuned by δ as shown in the coupling
scheme in Fig. 60(b).

Figure 60: (a) Schematic of the experimental setup. (b) Raman coupling scheme in the $F = 1$ manifold. (c) SOC dispersions $1 - 4$ are shown for $\Omega = 1, 1.5, 2, 2.7 \ E_R$, respectively, while $\delta = 500$ Hz. The arrows show the points of the dispersion which are coupled by the moving lattice. The color coding indicates the spin composition of the dressed states. (d) A typical experimental ToF image of the SOC condensate with the running lattice. This particular image is taken for $\Omega = 2.7 \ E_R$, $\delta = 250$ Hz, and $2 \ E_L$ moving lattice depth. Dashed rectangles $A$ and $C$ enclose the atoms that normally appear in the SOC ToF images while $B$ and $D$ represent those that have undergone the lattice two-photon transition.

The atoms that undergo the two-photon Raman process experience an internal state change as well as a $2\hbar k_R = 2\hbar \frac{2\pi}{\sqrt{2} \lambda_R}$ momentum transfer. The $\sqrt{2}$ appears due to the $90^\circ$ angle between the beams. The recoil energy associated with this two-photon process is $4E_R = 4\hbar^2 k_R^2$ in which $m$ is the $^{87}$Rb mass. For this setup, $E_R$ is approximately 1845 Hz.

The optical lattice is generated by interfering two perpendicular $\lambda_L = 1064$ nm laser beams at the position of the condensate. The moving lattice is realized by detuning the frequency of one of the laser beams. The characteristic momentum of the lattice is $2\hbar k_L = 2\hbar \frac{2\pi}{\sqrt{2} \lambda_L}$ and the corresponding two-photon recoil energy of the lattice is defined as $4E_L = 4\frac{\hbar^2 k_L^2}{2m}$. For our setup
$E_L \approx 1014$ Hz.

The condensate is adiabatically loaded into the SOC dressed states by ramping up the Raman beam intensities in $100$ ms while the detuning $\delta$ is 5 kHz. At such detunings almost all of the atoms are in the $|1, -1\rangle$ state. Once the intensities are ramped up, the detuning is adiabatically ramped down to the desired value.

The SOC strength $\Omega$ is chosen such that the quasimomentum difference between the two minima in the SOC dispersion matches the two-photon momentum of the optical lattice. Figure 60(c) shows the SOC dispersions for different coupling $\Omega$ demonstrating different quasimomenta associated with the lower band minima. The dispersion curve number 4 has a Raman coupling strength $\Omega = 2.7 E_R$. In this case, the spacing between the quasimomentum of the two minima matches the lattice momentum.

To reduce heating and atom loss in the SOC system, we introduce a non-zero detuning $\delta$. This causes an energy imbalance between the minima in the lower band of the dispersion as it can be seen in Fig. 60(c). The lattice velocity corresponding to the detuning is adjusted accordingly such that the two minima of the SOC dispersion are coupled. The black arrows highlight the spots on the dispersion that are coupled by the moving optical lattice. The color coding of the dispersion represents the spin composition of the states.

Since the condensate is adiabatically loaded into the SOC dressed states, it rests at the global minimum of the dispersion where the group velocity is zero and the energy is minimum. As a result, the optical lattice couples these atoms to the other minimum of the dispersion. It must be noted that the optical lattice two-photon transition conserves the atomic spin, and therefore it couples the atoms with the same spins in the two minima.

A typical experimental ToF image is shown in Fig. 60 (d). The image is taken using the Stern-Gerlach imaging procedure explained in Sec. 2.2.9. The atoms in $A$ and $C$ are the ones that are seen in a typical SOC ToF images without lattice coupling. They are in $|\uparrow\rangle$ and $|\downarrow\rangle$ atomic states respectively. Frames $B$ and $D$ enclose the atoms that are coupled by the optical lattice and therefore appear $2\hbar k_L$ apart from frames $A$ and $C$. 
In our experiments, we find the spin composition of the condensate for different moving lattice strengths and detunings. The spin compositions are obtained by finding the atom number imbalance between the upper $N_{|↑\rangle}$ and the lower $N_{|↓\rangle}$ frames in Fig. 60 (d). The results are shown in Fig. 61 (a-c) for three different detunings $\delta = 1000 \text{ Hz}, 500 \text{ Hz}, \text{ and } 250 \text{ Hz}.

In general, we observe a decrease in the spin composition as the moving lattice depth increases. It is also observed that the data spreads by increasing the moving lattice depth. The spread is more evident for the smaller detuning cases $\delta = 500, 250 \text{ Hz}$ found in Fig. 61(b, c).

![Figure 61](image)

Figure 61: Experimental results - Spin composition of the condensate vs. the moving lattice depth for (a) $\delta = 1000 \text{ Hz}$, (b) $\delta = 500 \text{ Hz}$, and (c) $\delta = 250 \text{ Hz}$. Every circle represents a single measurement. (a) and (b) contain approximately 60 measurements and (c) contains approximately 90 measurements.

The lattice couples quasimomentum state $k$ to $k \pm 2\hbar k_L$ while preserving the spin state. Depending on the detuning, the ground state of the SOC BEC lies at $k = \pm k_L$ because we chose the SOC strength such that the dispersion minima are separated by $2\hbar k_L$. The $k = \pm k_L$ states are
located close to the minima of the bare state dispersion relations as it can be seen in Fig. 62 (a).
The points labeled 1 through 6 are the states that are coupled via the SOC and the lattice. Note
that in the quasimomentum picture the Raman coupling does not change the quasimomentum.
Figure 62 (b) shows the oscillations between these bare states as the lattice is ramped up in 50 ms.
Notice that in this model, we assume that most of the atoms initially occupy the left parabola.
This problem can be viewed in the dressed states picture without missing any details as shown in
Fig. 62 (c) with the corresponding coupled states marked as 1’ through 6’. Note that in the dressed
states picture we do not see the Raman coupling \( \Omega \) anymore as the Raman coupling opens a band
gap where the two bare states cross. Similar to the bare state picture, the occupation probabilities
of the primed states can be calculated as a function of time. The results are presented in Fig. 62
(d).

Figure 62: (a) Six states in the bare state basis and the coupling between them is schematically
shown. (b) Occupation probability of the bare states as a function of time during the 50 ms lattice
potential ramp. A realistic detuning \( \delta \) is included in the simulation but it is not shown in the figure.
(c) The equivalent states in the dressed states basis of and the corresponding couplings between
them are presented. (d) Occupation probability of the dressed states as a function of time during
the lattice potential ramp.
We use this mathematical model to understand the reason behind the spin composition broadening observed in Fig. 61. The basic reason lies within the facts that: 1. The system is in a transient state and is oscillating between the eigenstates of the Hamiltonian as the lattice is ramped up. And, 2. The analysis shows that the oscillation is sensitive to small variations of the Raman coupling strength and the lattice height. Therefore, a slight noise in the experiment causes an extra phase difference, and we observe a spread in the results. The dependence of the spin composition \( \langle \sigma_z \rangle \) on the Raman coupling strength \( \Omega \) and lattice height \( V \) is presented in Fig. 63.

Figure 63: (a) Spin composition \( \langle \sigma_z \rangle \) as a function of time \( t \) for SOC strength \( \Omega = 2.7E_R \) and lattice strength \( V = 1E_R \). (b) \( \langle \sigma_z \rangle \) at \( t = 50 \) ms as a function of \( \Omega \) at \( V = 1E_R \). (c) \( \langle \sigma_z \rangle \) at \( t = 50 \) ms as a function of \( V \) for \( \Omega = 2.7E_R \).

An interesting question to ask would be why the lattice ramp is not adiabatic although the 50 ms ramp is much slower than the SOC Rabi cycle. The unusually long time scale of the system might be understood by considering the fact that an adiabatic process needs to be much slower than all the relevant dynamics of the system. In this case, the Raman beams are coupled to the lattice beams and therefore, being in the ground state of new dressed states requires the ramp to be slower than the combined Raman and lattice four-photon processes time scale. The theoretical analysis of the problem suggests that although the lattice is ramped up in 50 ms, it is not slow enough for an adiabatic process in this system. Figure 64 shows that an adiabatic process requires at least a 500 ms long lattice ramp with a \( 2E_R/s \) ramp rate.
Figure 64: Adiabatic lattice ramp rate occurs at much slower rates compared to Raman coupling time scale. Here, the SOC coupling strength $\Omega$ is $2.7 \, E_R$ and the lattice is ramped up with the rate of $2E_R/s$ for 500 ms.

Even though a 50 ms ramp is not long enough for an adiabatic process to take place, transient mixing of the two minima of the SOC dispersion can lead to a stripe phase although it is not the thermodynamic ground state of the system. This transient mixing can be observed in figures 62(b) and 62(d), which shows a 50/50 mixture between the 5$'$ and 2$'$ states. For example, as it can be seen in Fig. 63 (a), approximately 8 ms after the lattice ramp is started, we observe that the spin composition $\langle \sigma_z \rangle$ vanishes. At this point, we have a 50/50 mixture between the states 2$'$ and 5$'$ as it is shown in Fig. 62 (d). If the process is coherent, this means that the system is in the stripe phase at that moment. In a coherent Rabi process, the quantum state keeps oscillating between two or more states similar to what is shown in Figs. 62 (b) and 62 (d). Experimentally we can not resolve the individual oscillations in the curves due to slight noise, as previously discussed. In an incoherent process, the quantum states do not oscillate and just follow a thermal distribution after a short period and this is not what we observe in Fig. 61. In this figure, we see the spread of the data points at the end of the lattice ramp period which means that the oscillation between the quantum states is not damped after the 50 ms ramp.

### 3.4.3 Theoretical notes

In the following, we study the real-time response of a spin-orbit-coupled BEC when a lattice potential is gradually turned on. The optical lattice with a wave vector $k_L$ induces a first-order coupling
between the state with quasimomentum $k$ and $k \pm 2k_L$ states of the dispersion with the same spin state. Therefore, counting these states together with the pseudo-spin states we can write down a $(2 \times 3) \times (2 \times 3)$ Hamiltonian formulated as follows:

$$H = \begin{pmatrix}
(k - k_R + 2k_L)^2 & V & 0 & 0 & 0 & 0 \\
V & (k - k_R)^2 & V & \Omega^2 & 0 & 0 \\
0 & V & (k - k_R - 2k_L)^2 & 0 & \Omega^2 & 0 \\
0 & \Omega^2 & 0 & (k + k_R)^2 & V & 0 \\
0 & 0 & \Omega^2 & V & (k + k_R - 2k_L)^2 & V \\
0 & 0 & 0 & V & (k + k_R - 4k_L)^2 & \end{pmatrix}, \quad (3.26)$$

where the matrix basis 1–3 are for the spin up and 4–6 are for the spin down, $V$ is the lattice depth, $\Omega$ is the SOC strength, and $k_R$ is the Raman recoil momentum. The Hamiltonian is expressed in units of $E_R = \frac{\hbar^2 k_R^2}{2m}$ and the momentum is expressed in units of $\hbar k_R$. Thus, $k$ is expressed in units of $k_R$ and therefore $k_R = 1$ in Eq. 3.26. However, we leave $k_R$ as is in the equation for clarity. We consider a special case that

$$k_L = \sqrt{1 - \left(\frac{\Omega}{4}\right)^2}. \quad (3.27)$$

Equation 3.27 states that the lattice momentum matches the quasimomenta of the two minima in the SOC dispersion. In this case, the ground state of the BEC without the lattice ($V = 0$) has the quasimomentum

$$k = k_L. \quad (3.28)$$
To study how the ground state responds to the lattice, we rewrite the Hamiltonian by letting $k = k_L$ and $\Omega^2 = \sqrt{1 - k_L^2}$. Thus,

\[
H = \begin{pmatrix}
(3k_L - 1)^2 & V & 0 & 0 & 0 & 0 \\
V & (k_L - 1)^2 & V & \sqrt{1 - k_L^2} & 0 & 0 \\
0 & V & (k_L + 1)^2 & 0 & \sqrt{1 - k_L^2} & 0 \\
0 & \sqrt{1 - k_L^2} & 0 & (k_L + 1)^2 & V & 0 \\
0 & 0 & \sqrt{1 - k_L^2} & V & (k_L - 1)^2 & V \\
0 & 0 & 0 & 0 & V & (3k_L - 1)^2
\end{pmatrix}.
\]

(3.29)

In Fig. 62 (a) we label the six states of the Hamiltonian basis on the bare state spin branches and show the couplings between them. Note that the interactions are ignored for these simulations. The system starts at its ground state, which is a combination of $|2\rangle$ and $|4\rangle$ but mostly in $|2\rangle$, and oscillates mainly between $|2\rangle$ and $|5\rangle$ as the optical lattice is ramped up. The probability of the other four states is relatively small, but oscillations are present.

A different approach to this problem is to change the bare state basis to the dressed state picture which diagonalizes the SOC Hamiltonian. After a unitary transformation, we have

\[
H' = U^\dagger H U = H_0 + V \times H_1,
\]

(3.30)

with

\[
H_0 = \text{diag} \left( (3k_L - 1)^2, \ k_L^2 - 1, \ k_L^2 + 3, \ k_L^2 + 3, \ k_L^2 - 1, \ (3k_L - 1)^2 \right).
\]

(3.31)
and

\[
H_1 = \begin{pmatrix}
0 & \sqrt{\frac{1+k_L}{2}} & \sqrt{\frac{1-k_L}{2}} & 0 & 0 & 0 \\
\sqrt{\frac{1+k_L}{2}} & 0 & k_L & 0 & \sqrt{1-k_L^2} & 0 \\
\sqrt{\frac{1-k_L}{2}} & k_L & 0 & \sqrt{1-k_L^2} & 0 & 0 \\
0 & 0 & \sqrt{1-k_L^2} & 0 & -k_L & \sqrt{\frac{1-k_L}{2}} \\
0 & \sqrt{1-k_L^2} & 0 & -k_L & 0 & -\sqrt{\frac{1+k_L}{2}} \\
0 & 0 & 0 & \sqrt{\frac{1-k_L}{2}} & -\sqrt{\frac{1+k_L}{2}} & 0
\end{pmatrix}.
\]

In the basis used for Eq. (3.32), the couplings between states are induced only by the lattice potential \(V\). We label the 6 eigenstates of \(H_0\) in Fig. 62(c) and schematically show the couplings between them. In Fig. 62(d) we plot the same time evolution as in (b) but in the new basis. The system starts at the ground state \(|2\rangle\) and oscillates mainly between \(|2\rangle\) and \(|5\rangle\) with time. The second most populated states are \(|1\rangle\) and \(|6\rangle\) on the lower branch, and the least populated states are \(|3\rangle\) and \(|4\rangle\) on the higher branch. The state with lower energy gap from the ground state gets higher average population. The nearly degenerate state \(|5'\rangle\) thus has the largest oscillation amplitude compared with the other excited states.

We would like to consider realistic effects such as the external trapping potential and interactions. Therefore, we perform GP simulations to properly treat the problem. In Fig. 65 we plot the spatial density (left) and real-momentum (right) distributions of a BEC at \(t = 0\) (top panel) and after the 50 ms ramp (bottom panel). The results are comparable to Fig. 62 (b) (see the numbers denoting the corresponding states in Fig. 62(a)). This justifies the choice of six states in Eqs. (3.29) and (3.30). Note that the quasi-momentum of spin up and down states in Fig. 62(a) should shift by \(\pm k_R\) to the real-momentum space in the left panels Fig. 65. As a result, the three spin-up peaks take place at \(-k_L - k_R \sim -1.74\), \(k_L - k_R \sim -0.26\), and \(3k_L - k_R \sim 1.21\) and the three down-spin peaks do at \(-3k_L + k_R \sim -1.21\), \(-k_L + k_R \sim 0.26\), and \(k_L + k_R \sim 1.74\).
Figure 65: Top: Ground-state spatial density (left) and the real-momentum (right) distributions of a SOC BEC in a typical trap and with interactions. The wavefunction is obtained from the numerical GP equations. Bottom: same distributions at $t = 50$ ms after the lattice potential linearly increases to $V = E_R$. The numbers denote corresponding states as in Fig. 62(a).

Figure 66 (a) shows the spin polarization $\langle \sigma_z \rangle$ vs time using GP simulations. It is observed that the GPE results have a longer oscillation period than the single-particle approach. This is probably due to the interactions or the trapping potential. In Fig. 66 (b) we plot the population of the 6 largest momentum distribution peaks (as identified in the right bottom panel in Fig. 65) vs time. The results show similar physics to the single-particle results in Fig. 62 (b).
3.5 OBSERVATION OF THE FESHBACH RESONANCE BETWEEN $|1, +1\rangle$ AND $|2, -1\rangle$ STATES OF $^{87}$Rb

We have experimentally observed the Feshbach resonance between the two states $|1, +1\rangle$ and $|2, -1\rangle$ of $^{87}$Rb by creating a 50/50 mixture of the states while the magnetic field is far from the resonance and suddenly jumping the magnetic field to the final value and waiting for 50 ms for the losses to occur. The closer to the resonance the magnetic field is, the higher the loss rate would be, and therefore fewer atoms remain in the trap after the wait time.

Figure 67 shows the atom loss, measured after the wait time, versus the magnetic field. The error sources in this measurement are due to the initial magnetic field calibration based on spectroscopy of the $F = 1$ Zeeman levels as well as the reproducibility of the current servo system. Despite these systematic and statistical error sources, we observe a relatively good agreement between our measured value (9.14 G) and the reported value (9.1 G) by Hirano et. al. [2].
Formation of the elusive stripe phase depends on the miscibility between the two SOC atomic states. The presence of the Feshbach resonance between the states $|1, +1\rangle$ and $|2, -1\rangle$ can lead to the realization of the stripe phase. Based on the studies presented by Li et al. [111, 112], we evaluate the critical Raman coupling strength for different magnetic fields. The critical Raman coupling strength $\Omega_C$ is the maximum Raman coupling that allows the stripe phase as the ground state of the system. Together with the Raman detuning $\delta$, $\Omega_C$ indicates the region in the phase diagram in which the stripe phase can be observed (e.g. see [9, 111]). Our proposed method for the realization of the SOC using these atomic states can be found in Sec. 2.3. Another benefit of this method is its insensitivity to magnetic field fluctuations. This insensitivity is due to the fact the atomic states $|1, +1\rangle$ and $|2, -1\rangle$ have the same Zeeman energy shift up to the first order in the bias magnetic field $B$. 

Figure 67: Atom loss curve close to the Feshbach resonance. Each point represents one measurement. The red curve is a Gaussian fit with center position of 9.14 G and width of 0.095 G. The nominal magnetic field is calculated based on a measured magnetic field close to 10 G using RF spectroscopy. Then the smaller magnetic fields are found by linear extrapolation based on the electric current of the little bias coils. The linearly extrapolated magnetic field might be slightly different from the actual magnetic field due to a slight nonlinearity in the current servo system.
The magnetic dependence of the effective inter-species scattering length is

\[ a_{12}^{eff} = a_{bg} + \Delta a(B) = a_{bg} \left( 1 - \frac{\Delta B(B - B_0)}{(B - B_0)^2 + (\gamma_B/2)^2} \right), \]  

(3.33)

in which \( a_{bg} \) is the background scattering length which is 97.66 \( a_0 \), and \( B_0 \), \( \Delta B \), and \( \gamma_B \) are experimentally determined to be 9.104 G, 3 mG, and 13 mG respectively \[2\].

The critical Raman coupling \( \Omega_C \) can be obtained using

\[ \Omega_C = 2 \sqrt{\frac{\tilde{k}_R^2 + G_1(\tilde{k}_R^2 - 2G_2)}{G_1 + 2G_2}}, \]  

(3.34)

where \( \tilde{k}_R \) is half of the Raman momentum transfer and \( G_1 \) and \( G_2 \) are the interaction parameters

\[ G_1 = n(g + g_{12})/4 \]

\[ G_2 = n(g - g_{12})/4, \]

in which \( n \) is the average density and \( g = (g_{11} + g_{22})/2 \) \[111\]. Note that \( \tilde{k}_R \) is dimensionless and is set to 1 in Eq. (3.34). Using the same units the interaction coefficients \( G_1 \) and \( G_2 \) become dimensionless since \( g_{ij} \) is defined as \( \frac{4\pi a_{ij}}{k_R^2} \) in which \( a_{ij} \) and \( k_R \) are in SI units.

The interspecies scattering length \( a_{12} \) and the critical coupling strength \( \Omega_C \) as a function of the applied magnetic field are presented in Figs. 68 (a) and (b) respectively.
Figure 68: (a) Magnetic field dependence of the interspecies scattering length between the atomic states $|1, +1\rangle$ and $|2, -1\rangle$ is evaluated based on Eq. (3.33) based on the parameters estimated in [2]. (b) The critical Raman coupling strength $\Omega_C$ is calculated based on Eq. (3.34) for magnetic fields close to the Feshnach resonance with the scattering lengths evaluated using Eq. (3.33). This value is maximum at $B \approx 9.12$ G.

As shown in Fig. 68 (b), $\Omega_C$ is large enough to be experimentally accessible for a range of magnetic fields. From the experimental point of view, calibrating the Raman coupling strength is easier for larger couplings. This is because the calibrations are done by measuring the Rabi frequency of the system, and the oscillations need to stay fast compared to the decoherence time. However, $\Omega_C$ is larger in the vicinity of the Feshbach resonance where the atom loss rate, heating, and decoherence are not negligible. Therefore, choosing the "right" magnetic field for this system is a battle between decoherence/loss and a reasonable Rabi coupling strength.

In the following, we present the experimental results that show the dynamics of the BEC (without SOC) for different magnetic fields (close to the Feshbach resonance) and wait times. Here, we study how changing the scattering length affects the dynamics of the system. The system is prepared by producing a 50/50 mixture of the atomic states $|1, +1\rangle$ and $|2, -1\rangle$ at approximately a 10 G bias magnetic field and then quickly changing the magnetic field to a final value close to the
Feshbach resonance (see Fig. 67). After changing the magnetic field to the final value, we wait for a variable amount of time to study the dynamics of the system. The results are shown in Fig. 69. Notice the clear difference between the 50 ms cases on opposite sides of Feshbach resonance in Fig. 69. The image on the right hand side of the Feshbach resonance \((B = 9.2 \, \text{G} \text{ and } t = 50 \, \text{ms})\) shows a squished \(|2, -1\rangle\) state (top) while the \(|2, -1\rangle\) state in the image on the left hand side of the resonance \((B = 9.075 \, \text{G} \text{ and } t = 50 \, \text{ms})\) is extended. This demonstrates that the ball-shell dynamics of the system is modified and this is due to the change in the miscibility of the system. These results are compatible with the results in ref. [2] proving that the scattering length is noticeably modified due to the Feshbach resonance.
Figure 69: ToF images of the Ball-shell structure in a 50/50 mixture of $|1, +1\rangle$ and $|2, -1\rangle$ in the vicinity of the Feshbach resonance. The atomic clouds shown on the top and the bottom of each image are in the $|2, -1\rangle$ and $|1, +1\rangle$ atomic states, respectively.
3.6 CONCLUSION

In this chapter, first, we presented the methods and the results for the excitation spectrum of a SOC BEC. We showed the experimental evidence for the existence of a roton-like feature in the collective excitation spectrum which can be softened using the laser frequency. We found that for the parameters of our system, this softening stops at a finite energy and is symmetric under a sign change of the Raman detuning. Next, using a shallow moving barrier that is swept through the BEC, we observed a directional dependence in the spin composition indicative of the existence of the roton-like mode. Next, we studied the 1D expansion of SOC BEC, and we found that a fast enough expansion will result in the occupation of the negative effective mass region of the dispersion. We did not observe a particular asymptotic expansion velocity. Instead, the stopping seems to be a dynamical effect with dependence on the initial conditions of the problem. However, we could confirm that the observed momentum exceeds the momentum of the inflection point contrary to the previous lattice experiments. Next, we presented our experimental results for a SOC system that is combined with a weak momentum matched lattice. The lattice momentum is such that it mixes the states of the two minima of the SOC dispersion potentially leading to the realization of the supersolid phase. The theoretical analysis suggests that a transient supersolid phase might be observed. In the end, we provide experimental evidence for the observation of the known Feshbach resonance that can be employed to facilitate the realization of the stripe phase.
CHAPTER 4

SPIN-MOMENTUM COUPLED BOSE-EINSTEIN CONDENSATES WITH LATTICE BAND PSEUDOSPINS

4.1 INTRODUCTION

Spin-momentum coupling (SMC), commonly called spin-orbit coupling, is a crucial ingredient for many important condensed matter phenomena such as topological insulator physics, topological superconductivity, spin Hall effects, etc. [14,15,113]. In this chapter, we introduce a novel alternative for Spin-Momentum Coupling (SMC) in which we do not use an atomic internal state as the "spin" state. Traditionally, Zeeman sublevels of hyperfine states are used as the pseudo-spin states similar to the material presented in Chapter 3. Ultracold atoms in optical lattice potentials have other types of degrees of freedom which can also be used to define pseudospins [8,114]. A natural and important question is whether such new types of pseudospins can be employed to generate SMC.

The work that is presented in this chapter is based on our publication “Spin-momentum coupled Bose-Einstein condensates with lattice band pseudospins”, Nat. Comm., 7, 10867 (2016). In this work, I led the data acquisition and analysis processes. Dr. Chunlei Qu and Prof. Chuanwei Zhang performed the theoretical calculations. All authors contributed in writing the manuscript.

In optical lattices filled with ultracold atoms, $s$- and $p$-orbital bands are separated by a large energy gap and can be defined as two pseudospin states. One significant difference between hyperfine state pseudospins and lattice band pseudospins lies in the energy dispersion of spin-up and spin-down orientations: the dispersion relations are the same for hyperfine state pseudospins, while they are inverted for lattice band pseudospins. It is well known from topological insulators and superconductor physics that inverted band dispersions, together with SMC, play a central role.
for topological properties of materials [31, 32, 115]. Therefore, it is natural to expect that the inverted band pseudospins, when coupled with the lattice momentum, may lead to interesting topological phenomena in cold atomic optical lattices. Recent experiments with shaken optical lattices (i.e. lattices in which the lattice sites are periodically in time shifted back and forth [116]) have realized a simple coupling (Ωσₓ coupling, where Ω is the coupling strength and σₓ a Pauli matrix) between s- and p-band pseudospins, analogous to Rabi coupling between two regular spins [117]. However, for the exploration of exotic phenomena in optical lattice systems, such as Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) phases [34, 35] and Majorana fermions [115], SMC with s- and p-band pseudospins is highly desirable [36–39].

In our experiments we realize such s-p band SMC for a Bose-Einstein condensate (BEC) using a weak moving lattice to generate Raman coupling between s- and p-band pseudospins of a static lattice [118]. The moving lattice acts as a periodic driving field [119–124] and has previously been used to generate an effective magnetic field in the lowest s-band of a tilted optical lattice [125,126]. In our experiment, the driving frequency of the moving lattice is chosen close to the energy gap between s- and p-bands at zero quasimomentum, leading to a series of hybrid s-p Floquet-Bloch (FB) band structures. FB band structures in optical lattices give rise to interesting and important phenomena in cold atoms and solids [127,128], as is evidenced by the recent experimental realization of a topological Haldane model in a shaken honeycomb optical lattice [129] and the observation of FB states on the surface of a topological insulator [130]. We show that the moving lattice generates two types of coupling between s- and p-band pseudospins: a momentum-independent Rabi coupling (Ωσₓ) and SMC (ασₓ sin(qₓd), where qₓ is the quasimomentum and d the lattice period), with strengths of the same order. The coexistence of these two types of coupling leads to asymmetric FB band dispersions [131]. We investigate the FB band structures by measuring the quasimomentum of the BEC. The initial phase of the moving lattice plays a significant role in the Floquet dynamics [122], the effects of which are explored through a quantum quench induced dynamical coupling of the FB bands. Results are compared to theoretical predictions from a simple two-band model and from numerical simulations of the Gross-Pitaevskii (GP) equation.
4.2 EXPERIMENTAL SETUP AND THEORETICAL FORMALISM

To generate the \( s\)-\( p \) band SMC and FB band structures, we begin with a \(^{87}\text{Rb} \) BEC composed of approximately \( 5 \times 10^4 \) atoms confined in a crossed dipole trap. A static lattice is generated by two perpendicular laser beams with wavelength \( \lambda \approx 810 \) nm intersecting at the position of the BEC, as schematically shown in Fig. 70(a). The harmonic trap frequencies due to the envelope of the static lattice beams and the crossed dipole trap are \( (\omega_x, \omega_y, \omega_z) = 2\pi \times (41, 159, 115) \) Hz, where \( e_x \) points along the lattice, \( e_y \) is the horizontal transverse direction, and \( e_z \) is the vertical direction. A weak moving lattice with the same lattice period as the static lattice, \( d = \pi / k_L \) where \( k_L = \sqrt{2\pi / \lambda} \), is then overlaid with the static lattice (Fig. 70(b)). The moving lattice beams are approximately 180 MHz detuned from the static lattice. A small frequency difference \( \Delta \omega \) between the two moving lattice beams determines the velocity of the lattice according to \( v_{\text{lattice}} = \Delta \omega / 2k_L \). To induce \( s\)-\( p \) orbital band coupling, \( |\Delta \omega| \) is chosen close to the energy gap \( E_{sp} \) between the \( s\) - and \( p\)-bands of the static lattice at quasimomentum \( q_x = 0 \). Figure 70 (c) illustrates the total lattice potential as a function of time. The dashed line on the left panel of Fig. 70 (c) traces the maximum optical potentials in different times. The trace is not spatially symmetric, and therefore the potential breaks the parity.
Figure 70: Experimental setup and schematic lattice illustration. (a) Experimental arrangement. The crossed dipole trap beams propagate in the $e_x$ and $e_z$ directions. The static and moving lattices have overlapping beams propagating along $e_x + e_y$ and $-e_x + e_y$. The static lattice is generated using the red beams with laser frequency $\omega$, and moving lattice is generated using the blue beams with laser frequencies $\omega'$ and $\omega' + \delta \omega$. (b) Lattice potentials along the $e_x$ direction. The lattice period $d$ is identical for the static lattice $V_0$ and the moving lattice $V_0'$. The initial offset between lattice sites of the static and moving lattice, $\Delta x$, is dependent on the initial phase $\phi_0$ between the two lattices. (c) illustrates the lattice beam intensities as a function of space and time.

One outstanding feature of the coupling scheme employed in these experiments is the asymmetry of the effective $s$-$p$ FB bands, which exhibit a local minimum located at a finite quasimomentum $q_x \neq 0$. The direction in which the minimum is shifted away from $q_x = 0$ is determined by the sign of $\Delta \omega$ (which determines the direction of motion of the moving lattice) and $|\Delta \omega| - E_{sp}$ (i.e. the detuning of the drive from the bandgap at $q_x = 0$). Before describing experimental results and a formal derivation of the band structure using Floquet theory [122, 123], we lay the groundwork by presenting a multi-photon resonance picture that provides intuitive insights (Fig. 71(a,b)). In this picture, one starts with the parabolic dispersion of a free atom in the absence of any external potentials. An optical lattice then induces $2n$-photon couplings (with $n$ being an integer number) between points of the dispersion relation due to absorption and stimulated emission processes. The
couplings are centered around pairs of points that fulfill conservation of energy and momentum. At these points, bandgaps open due to avoided crossings. Examples for possible couplings due to the static lattice (red arrows in Fig. 71(a)) and the moving lattice (blue arrows in Fig. 71(b)) and the associated bandgaps in the first Brillouin zone are shown in Fig. 71. Different coupling strengths lead to different sizes of bandgaps, which result in an asymmetric band structure.

In another pictorial way, the Floquet band structure for the time-periodic system can be constructed by creating multiple copies of the Bloch band structure of the static lattice that are offset in energy by $|\Delta \omega|$. The moving lattice couples the $p$-band and the shifted $s$-band (labelled by $s'$ in Fig. 71(b)) at points where the shifted $s$-band intersects the unshifted $p$-band. The gaps opened by the coupling can formally be calculated using Floquet theory.

![Figure 71: Experimental setup and schematic lattice illustration. (a,b) Illustration of the multiphoton processes for the driven lattice system and the corresponding FB band structure in the first Brillouin zone. The static lattice induces a large energy gap (I) through a 2-photon process and a small energy gap (II) through a 4-photon process. The moving lattice induces an energy gap when the $s$-band and the $p$-band are coupled through (III). A smaller energy gap is produced by a combination of the static and moving lattice (IV).](image)

Adiabatic loading of the BEC into an $s$-$p$ FB band is achieved by first ramping on the intensity of the static lattice, followed by adiabatically ramping on the moving lattice intensity. In this procedure, the initial relative phase between the two lattices, $\phi_0$ (Fig. 70(b)), becomes irrelevant and can effectively be set to zero. As we shall show in this chapter, if the moving lattice is suddenly
jumped on instead of adiabatically ramped on, this initial relative phase may manifest itself by drastically changing the dynamics of the system [122].

Theoretical work presented in this chapter is prepared by our collaborators Chuanwei Zhang and Chunlei Qu. We provide the theoretical background and introduction to the theoretical treatment utilized for modeling the experiments shown later in this chapter. We briefly discuss the two-state tight binding model leading to a time-dependent Hamiltonian. The derivation is followed by utilizing the standard Floquet theorem to obtain a time independent effective Hamiltonian. A two state tight binding analysis proves to capture the essence of our experiments as a significant atomic population is not excited to the higher bands.

4.2.1 Minimal two-band model

The dynamics of the BEC are governed by the full time-dependent GP equation, $i\hbar \frac{\partial}{\partial t} \psi(r,t) = [\mathcal{H}_0(t) + V_{\text{trap}} + V_{\text{int}}] \psi(r,t)$ where $V_{\text{trap}}$ and $V_{\text{int}}$ are the external trapping potential and the mean-field interaction, respectively. $\mathcal{H}_0(t)$ is the single-particle Hamiltonian,

$$\mathcal{H}_0(t) = \frac{p^2}{2m} + V_0 \cos^2(k_L x) + V' \cos^2(k_L x + \frac{\phi_0}{2} - \frac{\Delta \omega t}{2}),$$

(4.1)

where the second and the third terms describe the static and moving optical lattices, respectively, and $\phi_0$ is the initial relative phase between the two sets of lattices.

When the static lattice depth $V_0$ is large and when $|\Delta \omega|$ is close to the energy gap $E_{sp}$, higher orbital bands are not significantly populated in the driven process and the system is well described by a simple two-band tight-binding model [131]. Following the standard procedure in Floquet theory, we obtain the effective single-particle Hamiltonian

$$H_0^{\text{eff}} = \begin{pmatrix}
\epsilon_s(q_x) & \Delta_{sp} \\
\Delta_{sp}^* & \epsilon_p(q_x) - |\Delta \omega|
\end{pmatrix},$$

(4.2)

where

$$\Delta_{sp} = -i[\Omega - \alpha \sin(q_x d) + \beta \cos(q_x d)]e^{-i\phi_0}$$

(4.3)
is the coupling between \( s \) - and \( p \) -orbital bands that is induced by the moving lattice potential for \( \Delta_\omega > 0 \) (see Sec. 4.2.2 for more details), and \( \epsilon_s \) and \( \epsilon_p \) are the energy dispersions for the uncoupled orbital bands. The three coupling coefficients \( \Omega \), \( \alpha \) and \( \beta \) are given by 
\[
\Omega = \frac{V'}{4} \langle s_i | \sin(2kLx) | p_i \rangle, \\
\alpha = \frac{V'}{2} \langle s_i | \cos(2kLx) | p_{i+1} \rangle \quad \text{and} \quad \beta = \frac{V'}{2} \langle s_i | \sin(2kLx) | p_{i+1} \rangle,
\]
where \( |s_i\rangle \) and \( |p_i\rangle \) are the maximally localized Wannier orbital states in the \( i \) -th site. \( \Omega \) is the coupling between \( s \) - and \( p \) -orbital states in the same lattice site, while \( \alpha \) and \( \beta \) are the couplings between \( s \) - and \( p \)-orbital states of nearest neighbouring sites. SMC between \( s-p \) band pseudospins is represented by \( \alpha \sin(q_x d) \sigma_x \).

This derivation shows that the inversion symmetry of the FB band structure is broken due to the coexistence of couplings of different parities. When the moving lattice intensity is adiabatically ramped on, the quasimomentum of the BEC gradually shifts away from \( q_x = 0 \) in a definite direction following the hybrid band minimum. This is quite different from previous shaken lattice experiments [116] where the inversion symmetry of the band was preserved and the BEC could spontaneously choose either side of \( q_x = 0 \) as its ground state. In that case, the BEC needed to be accelerated to break the inversion symmetry. In our scheme, the position of the true minimum is uniquely determined by the moving velocity direction, moving lattice depth, and driving frequency.

This minimal two-band model captures the essential physics of the driven lattices as we have seen through the comparison of experimental measurements and theoretical values (see Figs. 72 and 73), demonstrating the observation of SMC between \( s-p \) band pseudospins. However, this model may deviate from the experiment when the modulated dynamics involve additional orbital bands or when the nonlinear interaction is strong such that the single-particle band structure will be renormalized by the interaction term.

### 4.2.2 Tight-binding model

We consider a BEC in a combination of a 1D static optical lattice and a moving lattice,

\[
V_{\text{static}}(x) = V_0 \cos^2(k_L x), \tag{4.4}
\]

\[
V(x, t) = V' \cos^2(k_L x + \frac{\phi_0}{2} - \frac{\Delta_\omega}{2} t). \tag{4.5}
\]
The single-particle Hamiltonian is given by

\[ H_0 = \int dx \Psi^\dagger(x) \left( \frac{p^2}{2m} + V_{\text{static}}(x) + V(x,t) \right) \Psi(x). \]

By expanding the wave function in terms of the two lowest orbital states of the static optical lattice and assuming that all other higher orbital bands are not relevant, we obtain

\[ \Psi(x) = \sum_{j,\alpha} b_{j,\alpha} w_{\alpha}(x-x_j), \tag{4.6} \]

where \( \alpha = s, p \) is the orbital index, \( j \) is the site index, and \( w_{\alpha}(x-x_j) \) is the Wannier function for orbital state \( \alpha \) localized at site \( x_j \) with the corresponding annihilation operator \( b_{j,\alpha} \). Substituting this wave function into the Hamiltonian, we obtain the tight binding model

\[ H = \sum_{(j,k)} \left( -t_s b_{j,s}^\dagger b_{k,s} + t_p b_{j,p}^\dagger b_{k,p} \right) + \sum_{j\alpha,k\beta} A_{j\alpha,k\beta}(t) b_{j\alpha}^\dagger b_{k\beta} + \sum_j \left( \epsilon_s b_{j,s}^\dagger b_{j,s} + \epsilon_p b_{j,p}^\dagger b_{j,p} \right) \tag{4.7} \]

\[ = \sum_{j,k} \begin{pmatrix} b_{j,s}^\dagger \\ b_{j,p}^\dagger \end{pmatrix}^T \begin{pmatrix} -t_s \delta_{(j,k)} + \epsilon_s \delta_{j,k} & 0 \\ 0 & t_p \delta_{(j,k)} + \epsilon_p \delta_{j,k} \end{pmatrix} \begin{pmatrix} b_{k,s} \\ b_{k,p} \end{pmatrix} + \begin{pmatrix} A_{j,s,k,s}(t) & A_{j,s,k,p}(t) \\ A_{j,p,k,s}(t) & A_{j,p,k,p}(t) \end{pmatrix} \]

where

\[ t_s = -\int dx w_{s}^*(x-x_j) \left( \frac{p^2}{2m} + V_{\text{static}}(x) \right) w_{s}(x-x_k), \]

\[ t_p = \int dx w_{p}^*(x-x_j) \left( \frac{p^2}{2m} + V_{\text{static}}(x) \right) w_{p}(x-x_k), \]

\[ \epsilon_s = \int dx w_{s}^*(x-x_j) \left( \frac{p^2}{2m} + V_{\text{static}}(x) \right) w_{s}(x-x_j), \]

\[ \epsilon_p = \int dx w_{p}^*(x-x_j) \left( \frac{p^2}{2m} + V_{\text{static}}(x) \right) w_{p}(x-x_j), \]
are the bare tunneling elements between the same orbital states on the adjacent sites \( |j - k| = 1 \) (i.e., \( \delta_{(j,k)} = 1 \)) and the on-site energies from the static optical lattice. The \( \langle \rangle \) in the summations indicates summing over the nearest neighbor sites. The moving lattice induces assisted or dressed tunneling elements between adjacent sites and also modifies the on-site energies:

\[
A_{js,ks}(t) = \int \! dx w_s^* (x-x_j) V(x,t) w_s(x-x_k),
\]

\[
A_{jp,kp}(t) = \int \! dx w_p^* (x-x_j) V(x,t) w_p(x-x_k),
\]

\[
A_{js,kp}(t) = \int \! dx w_s^* (x-x_j) V(x,t) w_p(x-x_k),
\]

\[
A_{jp,ks}(t) = \int \! dx w_p^* (x-x_j) V(x,t) w_s(x-x_k).
\]

**Time-independent effective Hamiltonian**

To obtain a time-independent effective Hamiltonian, we first eliminate the diagonal dressed terms by a unitary transformation

\[
U(t) = \exp \left[ \frac{i}{\hbar} \int_0^t dt' \sum_{j,k} \sum_{\alpha} A_{j\alpha,k\alpha}(t) b_{j\alpha}^\dagger b_{k\alpha} \right]. \tag{4.8}
\]

The rotated Hamiltonian is

\[
H'(t) = U(t)H(t)U^{-1}(t) - i\hbar U(t) \frac{\partial}{\partial t} U^{-1}(t). \tag{4.9}
\]

We make the approximation that the driving does not change the on-site energy and bare tunneling of the same orbital states:

\[
U(t)b_{j\alpha,k\beta}^\dagger b_{j\alpha,k\beta} U^{-1}(t) \approx b_{j\alpha,k\beta}^\dagger b_{j\alpha,k\beta}. \tag{4.10}
\]

We obtain

\[
H = \sum_{j,k} \begin{pmatrix} b_{js}^\dagger \\ b_{jp}^\dagger \end{pmatrix}^T \begin{pmatrix} E_s & A_{js,kp}(t) \\ A_{jp,ks}(t) & E_p \end{pmatrix} \begin{pmatrix} b_{ks} \\ b_{kp} \end{pmatrix},
\]
where $E_s = -t_s \delta_{j,k} + \epsilon_s \delta_{j,k}$ and $E_p = t_p \delta_{j,k} + \epsilon_p \delta_{j,k}$. The $s$-$p_x$ couplings induced by the moving lattice are

$$A_{js, kp}(t) = \int dx w_s^*(x - x_j) V(x, t) w_p(x - x_k)$$

$$= \langle s^j | V^2(\frac{\phi_0}{2} - \frac{\Delta \omega}{2} t)|p^k \rangle$$

$$= \frac{V'}{2} \langle s^j | [\cos(2k_L x) \cos(\phi_0 - \Delta \omega t)$$

$$- \sin(2k_L x) \sin(\phi_0 - \Delta \omega t)]|p^k \rangle.$$  \hspace{1cm} (4.11)

The moving lattice has the same wave vector as the static lattice so that

$$z_0 = \langle s^j | \cos(2k_L x)|p^j \rangle = 0,$$

$$z_1 = \langle s^j | \sin(2k_L x)|p^j \rangle \neq 0,$$

$$z_2 = \langle s^j | \cos(2k_L x)|p^{j+1} \rangle = -\langle s^j | \cos(2k_L x)|p^{j-1} \rangle \neq 0,$$

$$z_3 = \langle s^j | \sin(2k_L x)|p^{j+1} \rangle = +\langle s^j | \sin(2k_L x)|p^{j-1} \rangle \neq 0.$$  

After the Fourier transformation

$$b_{js} = \frac{1}{\sqrt{N}} \sum_{k_x} b_{k,s} e^{ik_x x_j},$$  \hspace{1cm} (4.12)

$$b_{jp} = \frac{1}{\sqrt{N}} \sum_{k_x} b_{k,p} e^{ik_x x_j},$$  \hspace{1cm} (4.13)

we obtain

$$H(k_x, t) = \begin{bmatrix} \epsilon_s - 2t_s \cos(k_x d) & \Pi_{sp}(t) \\ \Pi_{sp}^*(t) & \epsilon_p + 2t_p \cos(k_x d) \end{bmatrix},$$

where the $s$-$p$ coupling is given by

$$\Pi_{sp}(t) = i e^{i \phi_0} [\Omega + \alpha \sin(k_x d) + \beta \cos(k_x d)] e^{-i \Delta \omega t}$$

$$+ i e^{-i \phi_0} [-\Omega + \alpha \sin(k_x d) - \beta \cos(k_x d)] e^{i \Delta \omega t}.$$
and the coupling coefficients are defined as

\[
\begin{align*}
\Omega &= \frac{V'}{4} z_1 = \frac{V'}{4} \langle s^j | \sin(2k_L x) | p^j \rangle, \\
\alpha &= \frac{V'}{2} z_2 = \frac{V'}{2} \langle s^j | \cos(2k_L x) | p^{j+1} \rangle, \\
\beta &= \frac{V'}{2} z_3 = \frac{V'}{2} \langle s^j | \sin(2k_L x) | p^{j+1} \rangle.
\end{align*}
\]

For \( \Delta \omega > 0 \), we perform a unitary transformation followed by the rotating wave approximation,

\[
U_2(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i \Delta \omega t} \end{bmatrix}.
\]

This leads to the time-independent Hamiltonian

\[
H_{\text{eff}} = U_2(t) H(k_x, t) U_2^{-1}(t) - i\hbar U_2(t) \frac{\partial}{\partial t} U_2^{-1}(t) = \begin{bmatrix} \epsilon_s - 2t_s \cos(k_x d) & \Pi_{sp}(t)e^{-i\Delta \omega t} \\ \Pi_{sp}^*(t)e^{i\Delta \omega t} & \epsilon_p + 2t_p \cos(k_x d) - \hbar \Delta \omega \end{bmatrix},
\]

which returns to our effective Hamiltonian Eq. 4.2 where we have \( \Delta_{sp} = -i[\Omega - \alpha \sin(q_x d) + \beta \cos(q_x d)]e^{-i\phi_0} \). For \( \Delta \omega < 0 \), similar procedures result \( \Delta_{sp} = i[\Omega + \alpha \sin(q_x d) + \beta \cos(q_x d)]e^{i\phi_0} \).

Note that \( \beta \) is usually much smaller than \( \Omega \) and \( \alpha \), however we keep it for completeness.

### 4.3 EXPERIMENTAL RESULTS

In the following, we provide our experimental results followed by a few considerations for performing the following experiments including the condensate lifetimes and lattice depth calibration procedures.

#### 4.3.1 Dependence of quasimomentum on driving frequency

Figure 72(a) shows the measurement of the band minimum, \( q_{\text{min}} \), for different driving frequencies, \( \Delta \omega \), after adiabatically loading a BEC into a FB band. The driving frequencies are chosen such
that $\hbar \Delta \omega$ lies in the gap at $q_x = 0$ between the $p$-band ($4.64 \, E_R$, where $E_R = \hbar^2 k_L^2 / 2m = \hbar \times 1749.5$ Hz) and the $d$-band ($5.44 \, E_R$). After adiabatically loading a BEC into a FB band, all lasers are switched off and the BEC is imaged after 14 ms time-of-flight (TOF). In the experimental images clouds with three different kinetic momenta are seen: $q_{\text{min}}$ and $q_{\text{min}} \pm 2k_L$. The kinetic momentum of the middle component, $q_{\text{min}}$, is equal to the quasimomentum of the BEC in the hybrid $s$-$p$ band. Therefore the quasimomentum can be obtained by measuring the position of the three components in the TOF images [132]. Each data point in Fig. 72(a) is an average over five iterations of the measurement. A shift of the quasimomentum is detected that decreases with increasing driving frequency (Fig. 72(a)) as the coupling between the $p$-band and shifted $s$-band becomes weaker. The observed shift indicates a shift of the minimum of the upper hybrid band (Fig. 72(b)) into which the BEC is adiabatically loaded. The solid line in Fig. 72(a) shows $q_{\text{min}}$ calculated from a simple two-band model (see below) and is in reasonable agreement with the data. The symbols are the results from solution of the Schrödinger equation (squares) and the GP equation (stars) with finite nonlinear interaction strength. The periodically driven dynamics are simulated using the time-dependent GP equation for a two dimensional system with the same geometry as the experiment. The width of the BEC is $\sim 10 \, \mu$m. We see that the interaction could modify the single-particle results. Since the atom loss is usually large in the experiment, the numerical results are not intended for direct comparison with the measurements. See Methods for some samples of the GP simulations.
Figure 72: Effects of the driving frequency. (a) Band minimum $q_{\text{min}}$ for the upper hybrid band vs. driving frequency $\Delta\omega$. The depth of the moving lattice is $1\ E_R$. The filled circles are experimental measurements with standard deviation errorbars. The black line shows the theoretical prediction of a two-band model. The squares and stars are the results of numerical simulations of the Schrödinger equation and the GP equation, respectively. (b) Calculated upper hybrid $s$-$p$ FB band structure for different driving frequencies $\Delta\omega = 4.99\ E_R$, $5.1\ E_R$ and $5.22\ E_R$ from top to bottom. The lowest (black) curve is the $s$ orbital band without the presence of the driving field.

4.3.2 Effect of the moving lattice depth

Figure 73 presents a complementary data set for which the driving frequency is set to a constant value with $|\Delta\omega| < E_{sp}$ (Fig. 73a) or $|\Delta\omega| > E_{sp}$ (Fig. 73c) and the quasimomentum is determined for various depths of the moving lattice. The sign of $\Delta\omega$ determines the direction of motion of the moving lattice. For $|\Delta\omega| < E_{sp}$ the BEC resides in the lower hybrid $s$-$p$ FB band (Fig. 73(b)) while for $|\Delta\omega| > E_{sp}$ it is in the upper hybrid band (Fig. 73(d)). This leads to a shift of the $q_{\text{min}}$ into opposite directions for the two cases. For a given driving frequency, the coupling of the two bands is stronger for larger driving field strength (i.e. larger depth of the moving lattice) so that the BEC is shifted to a larger absolute value of quasimomentum.
Figure 73: Effects of the driving strength. Band minimum $q_{\text{min}}$ vs. driving field strength $V'$ for different driving frequencies of (a) $|\Delta\omega| = 2.92E_R$ and (c) $|\Delta\omega| = 5.21E_R$. The red points are experimental data with standard deviation errorbars, and the solid lines are the theoretical predictions from a two-band model. $\text{sgn}(\Delta\omega)$ determines the direction of motion of the moving lattice. (b,d) Corresponding hybrid band structures for different driving field strengths $V'_x = 1.5E_R$, $0.75E_R$ and $0E_R$ (red, blue, and black curves respectively).

In Fig. 74, we present the results of GP simulations for different moving lattice depths. The modification of the band structure minimum can clearly be seen from the shift of the quasimomentum of the BEC during the driving process.

Figure 74: Plots of the evolution of the BEC momentum space density distribution as a function of $k_x$ for different evolution times. The moving lattice depth is ramped to (a) $V' = 0.25E_R$, (b) $V' = 0.5E_R$, (c) $V' = 1.0E_R$ in the first 60 ms and then keep this value for the following 40 ms. The white lines indicate the center of the small dipole oscillation from which the band minimum is determined. The driving frequency is $|\Delta\omega| = 5.21E_R$. 
4.3.3 Phase transition

Figure 73 (a) illustrates the dependence of the quasimomentum of the dispersion minimum on the moving lattice strength. We observe a first-order phase transition in the condensate by increasing the moving lattice strength for $|\Delta_\omega| = 2.92E_R$. Beyond approximately $1.7 \, E_R$ the absolute minimum changes. This is shown in Fig. 75 (a). In this figure, the curves represent the dispersion relations from the smallest moving lattice strengths to the largest. As it can be seen, the quasimomentum of the minimum switches from the right-hand side to the left-hand side. Figure 75 (b) demonstrates the experimental results shown using the black dots as well as the trace of the global minimum of the dispersions shown in 75 (a). We observe an excellent agreement between the theory and the experiments.

![Figure 73](image)

Figure 75: Observation of the first order phase transition in the condensate for moving lattice strengths larger than approximately $1.7 \, E_R$. (a) Shows the theoretical estimation of the dispersion relations starting from small moving lattice strengths to the larger strengths from the top to the bottom. (b) Shows the trace of the dispersion minima as a function of the moving lattice strength. The data points are the experimental results that are in excellent agreement with the theory.

4.3.4 Stability of the Floquet system

Floquet systems such as the one in our experiment are described by quasienergy bands. They do not have a thermodynamic ground state, and in the presence of many-body interactions their stability can be affected by a variety of factors [133–135]. Resonance induced collective excitations
and modulational instabilities can lead to losses [136]. A prominent feature of band inversion is the fact that hybrid bands can transition between stable and modulationally unstable structures. As shown in Fig. 76(a), when the driving frequency approaches the dip $\alpha$ from the left, the lower hybrid band evolves from a globally stable structure to a locally stable structure and eventually enters the globally unstable region at the quasimomentum where BEC mainly resides (Fig. 76(b)). The instability causes excitations and heating of the BEC. Experimentally, we study the stability of the system by determining the number of condensed atoms left in the trap after the static and the moving lattices are successively and adiabatically ramped on. As the instabilities cause heating, the heated atoms are evaporated out of the trap. Therefore absorption imaging reveals an atom loss from the BEC as shown in Fig. 76. The dips $\alpha$, $\beta$, and $\gamma$ in Fig. 76(a) occur when the driving frequency is chosen such that it leads to a coupling close to the Bloch bands $p$, $d$ and $f$ of the static lattice at $q_x = 0$ respectively.
Figure 76: Stability of the Floquet system. (a) Number of atoms remaining in trap after adiabatically loading a BEC into the FB band, normalized to initial atom number determined from independent experimental runs. The static lattice is ramped on to $5.47 \, E_R$ in 200 ms. Then the moving lattice is ramped on to a depth of $V' = 0.5 \, E_R$ in 60 ms. The dips $\alpha$, $\beta$, and $\gamma$ occur close to the Bloch bands $p$, $d$, and $f$. Panels (b), (c), and (d) show the TOF images and the effective band structures for data points labeled (b), (c), and (d) in panel (a). The arrows show where the BEC is situated after the moving lattice is adiabatically ramped on.

The results presented in Fig. 76 are generated by normalizing the atom number before or after every measurement. This has been done by switching the moving lattice detuning to a constant value away from the lattice band edges to keep track of the atom number fluctuations. The reason we had these fluctuations is due to the room temperature drifts. The air duct guiding the air to the back of the room may have caused temperature drifts in the dipole beam intensity controller. Normally, the dipole trap is not very sensitive to the room temperature because final dipole intensity is such that it does not directly cut into the condensate. However, for these experiments, we used less atom number to reduce the heating in the cloud.
After we realized the connection between the room temperature and the atom number, we disconnected the air duct and made a temperature stabilizer for the dipole beam intensity controller. More details about the intensity controller can be found in section 2.6. The non-calibrated atom numbers versus the moving lattice velocity is presented in Fig. 77. The atom number drifts washes out the effect almost completely. The results clearly show the importance of the atom number calibrations during these measurements.

![Graphs showing non-calibrated BEC stability results](image)

**Figure 77:** Non-calibrated BEC stability results showing the atom number versus the moving lattice velocity. (a) and (b) correspond the loss curves shown in Fig. 76 (a) left and right respectively.

### 4.3.5 Quench dynamics

Since a Floquet system is generated by a time-periodic Hamiltonian, an important question concerns the role of the initial phase of the driving field [122]. For the system considered in this work, this phase determines the relative positions between the moving and static lattice sites. Though the relative phase does not change the effective band structure (Eq. 4.2), and thus the time-averaged dynamics, it can play a crucial role in the micromotion of the BEC. To demonstrate the effect of the initial relative phase, we study the oscillations in the population of the momentum components $k_x = 0, \pm 2k_L$ after a quantum quench. Figure 78 presents such quench dynamics after adiabatically ramping on the static lattice to $5.47 E_R$ followed by a sudden jump on of the moving lattice to $V_x' = 1E_R$ with an on-resonant driving frequency $|\Delta \omega| = E_{sp}$. Fig. 78(a,b) show the effective band
structure before (a) and after (b) the moving lattice is jumped on. The jump projects the BEC onto the new band structure and the BEC acquires components in the new lower and upper hybrid band. Subsequently these components beat against each other. We focus on the evolution during the first 3 ms, during which the BEC mainly stays at \( q_x = 0 \) without significant dipole motion in the hybrid bands. The TOF images consist of three momentum populations as shown in the two examples in Fig. 78(c), which are both taken after 0.5 ms of evolution. In our experiments, the initial phase \( \phi_0 \) between the static and the moving lattice is uncontrolled and is different in each repetition of the experiment. This leads to different population dynamics in each experimental iteration and explains the differences between the two images in Fig. 78(c). The population dynamics measured in ten subsequent experimental iterations are represented by the rectangles in Fig. 78(d, e). For each measurement time, the height of the rectangle indicates the spread in the data. The shaded areas represent the result of numerical GP simulations for a homogeneous spread of relative phases.

We find a strong correlation between the experimental spread and the spread predicted by the numerics (Fig. 78(d, e)). Numerical results calculated for the particular value \( \phi_0 = 0 \) are shown by the solid line in Fig. 78(d, e). They reveal that for a fixed initial phase there are oscillations on two different time scales. The fast oscillations (with period \( T \approx 0.12 \text{ ms} \)) corresponds to the micromotion of particles under the high-frequency periodic driving, whereas the slow oscillation (with \( T \approx 1.75 \text{ ms} \)) corresponds to the time averaged effective Rabi oscillations between the two hybrid FB bands. For longer holding times, the periodicity is slightly broken due to a small dipole motion. Figure 78(f) shows the correlation between numerical and experimental spread for the data in (d) and (e). The straight line is a fit showing the correlation trend. A linear correlation of \( \rho_{X,Y} = 0.64 \) is achieved, where \( \rho_{X,Y} \) is the Pearson product-moment correlation coefficient defined by \( \rho_{X,Y} = \frac{E[(X-E[X])(Y-E[Y])]}{\sigma_X \sigma_Y} \). Here \( E \) is the expectation value, and \( \sigma \) is the standard deviation.
Figure 78: Quench dynamics after suddenly jumping on the coupling between the $s$ and $p$ band. Band structure before (a) and after (b) jumping on the coupling between the $s$ and $p$ band. (c) Two examples for experimental images taken 0.5 ms after the quench. The difference between the images is due to the relative phase between the static and the moving lattice. The experimental data for panels (d) and (e) are extracted from such images by counting the atom number in the dashed boxes. They indicate the $-2\hbar k$ (left rectangle), 0 (middle rectangle) and $+2\hbar k$ component (right rectangle). Panels (d) and (e) represent the normalized population of the momentum component $-2\hbar k$ and $0 \hbar k$ respectively. The spread of the experimental data is indicated by the height of the rectangles. The shaded areas are the results of numerical GP simulations calculated for a uniform spread of phases $\phi_0$. The black curves show the numerical result for $\phi_0 = 0$. Panel (f) presents the correlation between numerical and experimental spread for the data from (d) and (e). The solid line is a fit to the scatter.

4.3.6 Experimental considerations

In the next few paragraphs, we study the lifetime of the condensate in the static lattice generated using 810 nm laser beams. A reasonable concern about using laser beams close to the $^{87}$Rb D1 and D2 lines is the spontaneous emission and therefore heating and losing particles fast compared to the experiment time scales.
Figure 79: Condensate lifetimes in the static lattice. (a) and (b) show the atom numbers in arbitrary units for $5 E_R$ and $6.5 E_R$ deep static lattices respectively.

The results show 4.1 s and 2.1 s lifetimes for static lattice strengths $5 E_R$ and $6.5 E_R$ respectively. Both of these lifetimes are long enough to allow for applying the moving lattice on the top of the static lattice. However, the presence of the moving lattice simultaneously with the static lattice significantly reduces the lifetime to hundreds of milliseconds. However, hundreds of milliseconds is still long enough for the dynamics that occur in our experiments.

Calibrating the lattice strength is critical for making the experiments reproducible. We use Rabi oscillations for calibrating the lattice strengths. We suddenly turn on a moving lattice with detuning $\delta = 4E_R$ to measure the Rabi oscillations between the $s$ and $p$ bands edges at the boundary of the Brillouin zone. The oscillation frequency between the the bands gives the band gap energy. Sample TOF images of a Rabi frequency measurement is presented in Fig 80. The images are rotated by $90^\circ$ for convenience.

Figure 80: Sample TOF images of a full Rabi oscillation between the $s$ and the $p$ bands is presented.
Figure 81 (a) shows the Bloch bands for different lattice depths. The band gap increases in size by increasing the lattice depth as shown in Fig 81 (b) with the blue curve. The band gap is approximately 50% of the lattice strength for small lattice depths up to $5E_R$ lattice strength. For larger lattice depths the difference becomes more significant, and we need to compensate for that in our calibration procedure. We mostly use $5.47E_R$ for the static lattice depth in our experiments and ratio between the lattice depth and the band gap is approximately 2.06. Thus, we use 2.06 for calibrating our lattice depth after finding the Rabi frequency between the $s$ and $p$ band.

![Figure 81: Dependence of the Bloch bands on the optical lattice depth. (a) Illustrates the Bloch bands for optical lattice depths 0-7 $E_R$. Black corresponds to 0 $E_R$. Red and Cyan correspond to $p$ and $s$ bands for 7 $E_R$ optical lattice depth. (b) The blue curve shows the band gap at the Brillouin zone edge as a function of the lattice height. The red line represents $y = \frac{1}{2}x$.](image)

The same calibration is employed for calibrating the moving lattice strength as both laser beams are at 810 nm, and they follow the same beam paths.

### 4.4 CONCLUSION

In this chapter, we have realized and characterized a new kind of SMC with lattice bands as pseudospins. This not only provides a powerful tool to control orbital states with a driving field, but also enriches the study of novel quantum matter using hybrid orbital bands. There are many directions that can be taken along this route, e.g., the engineering of similar SMC in higher dimensional systems involving different orbital bands, and quantitative analysis and measurements of the effects
of strong interactions on the effective bands. The realization of similar SMC for fermionic atoms such as $^6\text{Li}$ and $^{40}\text{K}$ with tunable interactions may open the door for exploring the exotic quantum matter. For example, for a spin-balanced Fermi gas loaded into such tilted s-p dressed bands, the Cooper pairs may acquire a finite center-of-mass momentum due to the broken inversion symmetry which provides a new route to search for the long-sought FFLO states [131]. Furthermore, the realized spin-orbit coupling with lattice bands as pseudo-spins may provide a new platform for the study of topological insulators and Majorana fermions [31,32,115].
CHAPTER 5

GENERATION AND DYNAMICS OF SOLITONS IN BOSE-EINSTEIN CONDENSATES

5.1 INTRODUCTION

Atomic Bose-Einstein condensates (BECs) offer an excellent testbed for the exploration of nonlinear structures relevant to multi-component nonlinear wave systems [137, 138]. To date, dark-bright (DB) solitary wave and related structures such as dark-dark solitary waves that have long been studied theoretically [139–145].

Nonlinear solitary waves were experimentally realized much earlier in nonlinear optics, including e.g. the observation of dark-bright solitary wave structures in [146, 147]. However, the flexibility of BECs enabled a wide variety of relevant studies initially motivated by the proposal of [148]. Specifically, the experimental realization of DBs [51] was followed by a series of experiments investigating the dynamics and properties of these features including in-trap oscillations of DBs, their spontaneous generation (e.g. via counterflow experiments) and their interactions both with other DBs and with external potential barriers [149–154].

Despite these efforts, only a limited number of types of solitons have been discovered and experimentally observed in recent years. In this chapter, we provide experimental evidence for the existence of novel exotic solitary waves in multicomponent BECs and study their dynamics in the trap.

These hydrodynamic experiments are rich in physics and in particular we present theoretical and experimental results regarding the Dark-Antidark (DAD) solitons.

From a theoretical standpoint, in a multi-component condensate, we can generally write the
time-dependent GP equation in the following form

\[
\left( \frac{p^2}{2m} + V_i(x) + \sum_{j=1}^{n} g_{ij} |\Psi_j(x,t)|^2 \right) \Psi_i(x,t) = i\hbar \partial_t \Psi_i(x,t)
\]  \hspace{1cm} (5.1)

in which \(1 < i, j < n\) and \(n\) in the number of species in the condensate. In Eq. 5.1, \(p\) is the momentum operator, \(V_i\) is the potential for the \(i\)th component, and \(\Psi_i\) is the wave-function of the \(i\)th component. What makes the solutions to this equation interesting is the collective interaction between the inter and intra-atomic species which depends on the scattering length between the atoms.

The miscibility/immiscibility of the system is key in the dynamics of a multi-component BEC. The miscibility of a two-component system is scaled by \(\gamma = a_{11}a_{22} - a_{12}^2\) in which \(a_{ij}\) is the scattering length between the \(i\) and \(j\) atomic species. In the context of repulsive interactions, the larger the quantity \(\gamma\) is, the more miscible the system will be. The miscibility/immiscibility of the system plays an important role in what family of solitons can or cannot be generated in the condensate.

### 5.2 DARK-ANTIDARK SOLITONS

#### 5.2.1 Introduction

Very recently, a novel type of solitons called "magnetic solitons" was proposed by Qu et al. [52]. Particularly, in contrast with other solitonic waves such as Dark-Bright (DB) solitons that are understood relatively well, magnetic solitons have not been realized in the lab. Given the limited number of experimental investigations in connection with magnetic solitons (in comparison to other soliton families, such as bright or dark solitons) such entities naturally are of considerable interest.

A complementary possibility recognized considerably earlier was that of dark-antidark solitary waves [155]. Antidark solitary waves are bright solitary waves on top of a finite background. Here, we will avoid calling the structures under investigation “magnetic”, as we do not a priori constrain the sum of the densities of the two components to equal that of a single component ground state, as in the settings of [52, 156]. We will show that the idea of complementary non-trivial components,
We provide experimental evidence for the existence of these solitonic states confirming and complementing the theoretical framework. Our example of experimental realization of the dark-antidark solitons in a BEC is straightforward to reproduce in the current experimental settings. We also report on long-lasting oscillations of the DAD solitons in the dipole trap proving unexpectedly long lifetimes for these states.

This section of my dissertation is based on our publication “Vector dark-antidark solitary waves in multicomponent Bose-Einstein condensates”, PRA, 94, 053617 (2016). I led the experimental efforts and the data analysis. The theoretical work presented in this section is performed by our collaborators Prof. I. Danaila and Prof. P. Kevrekidis providing the framework for our experimental results. All authors contributed in preparing the manuscript.

5.2.2 Experimental considerations

The reproducible observation of solitary waves in an elongated BEC requires precise control of various experimental parameters. One important aspect of these experiments is the bias magnetic field. Since the condensates are prepared using different atomic hyperfine states of $^{87}$Rb, they might have different magnetic moments and therefore react differently to the magnetic field gradients. For example, for observing dark-antidark solitons, we use a mixture of $|F,m_F\rangle = |2, -2\rangle$ and $|F,m_F\rangle = |1, -1\rangle$ hyperfine states, which have magnetic moments with opposite signs. Thus, a slight magnetic field gradient exerts opposite forces on the atoms in each state and can cause a spatial offset between them in the trap. During our experiments, special care was taken to make sure that this gradient is negligible. Figure 82 illustrates how small variations in the horizontal (along the $x-$axis) magnetic field gradient can cause separation of the atomic species. The experiments are performed by preparing a 50/50 mixture of the atoms in the $|1, -1\rangle$ and $|2, -2\rangle$ states and waiting for 15 s. This wait time gives enough time to the atoms in different states to rearrange themselves in the trap. The currents in the “little bias coils” in Fig. 82(a) are 8.965 A and 9.03 A, respectively. In this case, the estimated magnetic field gradient is 100 mG/m. The residual magnetic field gradient
in Fig. 82(b) is estimated to be less than 10 mG/m. These magnetic field gradients are estimated using

\[ m \omega_x^2 x_0 = \mu_B g_F m_F \frac{\partial B_x}{\partial x} \]  

(5.2)

in which \( x_0 \) is the shift in the trap center due to the magnetic field gradient, \( \mu_B \) is the Bohr magneton, \( g_F \) is the g-factor for the hyperfine state, \( m_F \) is the projected angular momentum, \( m \) is the atomic mass, and \( \omega_x \) is the trapping frequency.

Figure 82: Panel (a) presents the elongated cloud imaged in TOF with \( |2, -2\rangle \) appearing on the top, and \( |1, -1\rangle \) appearing on the bottom. The currents in the little bias coils one and two for this configuration are 8.965 A and 9.03 A, respectively. Panel (b) illustrates the image similar to (a) with little bias coil currents 8.855 A and 9.03 A for little bias one and two coils respectively.

It is worth noting that the curvature of the magnetic field plays a crucial role in the configuration of the BEC. Since the atomic states \( |2, -2\rangle \) and \( |1, -1\rangle \) experience opposite Zeeman shifts for the same magnetic field, they react differently to the curvature of the magnetic field. A curvature in the magnetic field causes spatially varying potentials with opposite signs for each atomic state. For example, this curvature can attract one atomic state towards the center of the trap or repel it to the sides. At the same time, the scattering lengths of the atomic states specify how the atoms are arranged in the trap (e.g. see [157–159]). For example, in the immiscible regime, an atomic
state with smaller scattering length concentrates at the center of the trap while the other atomic state with larger scattering length extends further out in the trap. The effect of the magnetic field curvature together with the effect of the inter- and intra-component scattering lengths specifies how the atoms are arranged in the trap.

The lifetime of the atomic species in the trap is measured for a 50 – 50 atomic mixture by holding the mixture in the trap for a variable time. The results are shown in Fig. 83. Figures 83 (a) and (b) represent the atom loss curves for the atomic species \(|1, -1\rangle\) and \(|2, -2\rangle\), respectively. Both species show relatively long lifetimes for our experimental parameters.

**Figure 83:** (a) and (b) illustrate the atom loss curves as a function of hold time in the trap for \(|1, -1\rangle\) and \(|2, -2\rangle\) atomic states, respectively.

### 5.2.3 Experimental Results

To motivate our discussion, we begin by presenting experimental evidence for the existence, stability and dynamics of dark-antidark solitary waves. A ToF image of a dark-antidark soliton is shown in Fig. 84(a).
Figure 84: Experimental realization of dark-antidark solitary waves. (a) Absorption images (upper two panels) and corresponding integrated cross sections (lower two panels) of a dark-antidark solitary wave. The dark soliton component resides in a cloud of $|F, m_F \rangle = |2, -2 \rangle$ atoms (upper and third panel), while the bright component consists of atoms in the $|F, m_F \rangle = |1, -1 \rangle$ state (second and forth panel from top). (b) Experimentally observed oscillation of the dark-antidark solitary wave in the trap. The position is measured along the x-axis, i.e. along the weakly confining axis of the trap. The time is measured starting from the initial microwave pulse that creates the two-component mixture. The blue dots are experimental data, while the red line is a sine function fit to the data. (c) Comparison between a dark-bright soliton in a mixture of atoms in the $|F, m_F \rangle = |1, -1 \rangle$ and $|F, m_F \rangle = |1, 0 \rangle$ states (left image) and a dark-antidark structure in a mixture of atoms in the $|F, m_F \rangle = |2, -2 \rangle$ and $|F, m_F \rangle = |1, -1 \rangle$ states (right image).
In our experiments, we observe these features in two-component BECs confined in an elongated dipole trap. The experiments begin by creating a BEC of approximately $0.8 \times 10^6$ $^{87}$Rb atoms held in an optical trap with harmonic trap frequencies of $\omega_{x,y,z} = 2\pi \{1.4, 176, 174\}$ Hz, where $z$ is the direction of gravity. Evaporation in the dipole trap is continued until no thermal fraction is discernible. Initially, all atoms are in the $|F, m_F\rangle = |1, -1\rangle$ hyperfine state. Subsequently, a brief microwave pulse transfers a fraction of the atoms (approximately 50% for the case shown in Fig. 84) into the $|2, -2\rangle$ hyperfine state. The transfer occurs uniformly across the whole BEC.

The resulting two-component BEC can be described by two separate Gross-Pitaevskii equations that are only coupled by the intercomponent scattering length (see the theory section below). The intra- and inter-component scattering lengths for the two states are $a_{11} = 100.4a_0$, $a_{22} = 98.98a_0$, and $a_{12} = 98.98a_0$, where $a_{11}$ denotes the scattering of two atoms in the $|1, -1\rangle$ state, $a_{12}$ the scattering between an atom in the $|1, -1\rangle$ and $|2, -2\rangle$ state, and $a_{22}$ between two atoms in the $|2, -2\rangle$ state [160]; $a_0$ is the Bohr radius. Based on the standard miscibility argument [161–163], discussed above, this mixture is slightly miscible. However, the difference between $a_{11}$ and $a_{22}$ leads to a slight concentration of $|2, -2\rangle$ atoms towards the center of the cloud. When the mixture is held in the trap for approximately 10 sec or longer, the emergence of a dark-antidark solitary wave is observed as shown in Fig. 84. The solitary waves are imaged by suddenly switching off the trap and imaging the $|2, -2\rangle$ state after 10 ms of expansion and the $|1, -1\rangle$ state after 11 ms of expansion. The difference in the free-fall time separates the two images on the camera so that the two components appear below each other in the images. During all in-trap evolution leading to the soliton formation, the two components have been well overlapped vertically.

Repeating this procedure with well controlled experimental parameters, we observe that each iteration of the experimental run reliably produces a two-component BEC containing one dark-antidark solitary wave such as the one shown in Fig. 84. In all iterations, the dark soliton resides in the $|2, -2\rangle$ component and the antidark soliton is found in the $|1, -1\rangle$ component.

It is remarkable that these features emerge quite “naturally” in our experiments without any dedicated wavefunction engineering [149,152] or phase imprinting [51]. Furthermore, these features
are very long lived. We have observed their in-trap dynamics for up to 30 sec. For comparison, starting with a 50/50 mixture of the two components, we measure the lifetime of the $|2, -2\rangle$ component to be $\sim 13$ sec and that of $|1, -1\rangle$ component to be $\sim 15$ sec for our experimental parameters. The emergence of these solitary waves is rather insensitive to the exact mixture ratio of the components. Experimentally we tested and confirmed their existence in a variety of mixtures ranging from 30% of the atoms in the $|2, -2\rangle$ state and 70% in the $|1, -1\rangle$, to mixtures of 50% in the $|2, -2\rangle$ state and 50% in the $|1, -1\rangle$ state. In mixtures where the abundance of the $|2, -2\rangle$ component exceeded approximately 50%, no clear soliton formation was observed.

We have also repeated the experiment with a mixture of atoms in the $|1, -1\rangle$ and $|1, 0\rangle$ components. The scattering lengths for this mixture are $a_{11} = 100.4 a_0$, $a_{22} = 100.86 a_0$, and $a_{12} = 100.41 a_0$, where $a_{11}$ now denotes the scattering of two atoms in the $|1, -1\rangle$ state, $a_{12}$ the scattering between an atom in the $|1, -1\rangle$ and $|1, 0\rangle$ state, and $a_{22}$ between two atoms in the $|1, 0\rangle$ state [160]. This mixture is closer to the miscibility-immiscibility threshold. Following an analogous procedure as described above, no formation of dark-antidark solitary waves was observed. Instead, dark-bright solitons were formed. The result is shown in Fig. 85. The ToF image shows the condensate after 15 s wait time in the trap followed by the separation of the spin states using a vertical 3 ms Stern-Gerlach kick. Other in-trap wait times show the same result.

![Figure 85](image.png)

Figure 85: DB solitons are formed using the $|1, -1\rangle$ and $|1, 0\rangle$ atomic states. The experimental procedure used for this figure is similar to that of dark-antidark solitons. For this particular image, there is 15 s wait time inside the trap.

A comparison between a dark-bright soliton and a dark-antidark soliton is shown in Fig. 84(c), showcasing their different structure. This emphasizes the important role that the miscibility of the
components plays for the generation of a non-zero background in the second (bright) component, as has also been highlighted in [52].

We note here an important open issue that we will not fully address in our numerical investigation that follows, namely that of experimentally identifying only dark-antidark states in the first of the above mixtures and only dark-bright ones in the second one. Our numerical results below will show that dark-antidark states exist up to the miscibility-immiscibility threshold (although as the latter limit is approached, they may be difficult to identify given their degeneration to dark solitons). On the other hand, the work of [164] suggests that dark-bright solitary waves exist on both sides of the miscibility-immiscibility transition. This combination indicates that a closer investigation of the proximity to the transition limit (possibly also involving further realistic effects such as genuinely 3d computations) may be necessary to further explore this feature. Also, as mentioned above, the effect of the magnetic field curvature can be essential in formation of the dark-antidark solitons as the field curvature can introduce an artificial miscibility/immiscibility to the system.

The long lifetimes and reproducible generation of dark-antidark solitary waves in a mixture of atoms in the $|2,-2\rangle$ and $|1,-1\rangle$ states allow us to observe their in-trap dynamics (Fig. 84(b)). We clearly detect a slow oscillation of the solitary wave along the weak axis of the trap. A fit of the data in Fig. 84(b) gives an oscillation period of approximately 5.6 sec. For comparison, the period of a dark soliton in a single-component BEC in the same trap is predicted to be 1 sec [165, 166]. Hence, the dark-antidark solitary waves are significantly slower. A similar trend has been found in [148,150], where it was seen that dark-bright solitons are slowed down when the number of atoms in the bright component is increased. The theory of "magnetic solitons" described in [52] assumes $a_{11} = a_{22}$, which in our experiment is only approximately fulfilled. This theory predicts an oscillation period on the order of 9.8 sec, somewhat longer than that observed in the experiment. A quantitative comparison between experiment and theory, including the influence of the mixing ratio of the two components and the finite lifetime of the trapped atoms, will be left for future work. Here, these first observations of dark-antidark solitary waves serve as a motivation to investigate (chiefly
as a function of the inter-component scattering length) and generalize the underlying concepts of dark-antidark structures using numerical continuation studies and Bogolyubov-de Gennes analysis.

5.2.4 Theoretical/Numerical Results

The results of this section have been provided by our theory collaborators prof. Kevrekidis and prof. Danaila.

The atomic states forming the dark-antidark solitons have a complementary intensity profile in the two-components \((\psi_1(x, t), \psi_2(x, t))\) and are described by the two-component wave function

\[
\begin{pmatrix}
\psi_1(x, t) \\
\psi_2(x, t)
\end{pmatrix} = \sqrt{n} \begin{pmatrix}
\cos\left(\frac{\theta}{2}\right) e^{i\phi_1} \\
\sin\left(\frac{\theta}{2}\right) e^{i\phi_2}
\end{pmatrix}
\]

where \(\theta(x, t)\) characterizes the spatial distribution of the amplitude, \(n\) the total density and \(\phi\) the phase of each component. It is relevant to note that a related idea regarding the formulation of the multi-component nonlinear wave state was put forth, e.g., in the work of [156].

To capture the qualitative features of the states of interest, it will suffice to utilize a mean-field model in the form of the Gross-Pitaevskii equation. Upon suitable standard reductions [137, 138], the model can be transformed to its dimensionless version in the form:

\[
\begin{align*}
t \frac{\partial \psi_1}{\partial t} &= -\frac{1}{2} \Delta \psi_1 + V(x) \psi_1 + (g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2) \psi_1 \\
t \frac{\partial \psi_2}{\partial t} &= -\frac{1}{2} \Delta \psi_2 + V(x) \psi_2 + (g_{12}|\psi_1|^2 + g_{22}|\psi_2|^2) \psi_2
\end{align*}
\]

Here, the pseudo-spinor field is denoted by \((\psi_1, \psi_2)^T\) (where \(T\) is used for transpose), \(V(x) = \frac{\Omega^2}{2} x^2\) represents the parabolic trap, while \(g_{ij}\) are the interaction coefficients, proportional to the experimental scattering lengths mentioned above. In line with the analysis of [52], we will assume in what follows that \(g_{11} = g_{22} = g\), while \(0 \leq g_{12} \leq g\). Given that only the ratios of the scattering lengths matter, we will choose \(g = 1\), while \(0 \leq g_{12} < 1\), in order to be in the miscible regime, while preserving inter-component repulsion.

In our numerics, the stationary states \((\psi_1^{(0)}, \psi_2^{(0)})^T\) are identified by virtue of a fixed point
iteration (typically a Newton method) originally at \( g_{12} = 0 \), i.e., the limit where the two components are uncoupled. Then, parametric continuation is utilized in order to follow the configuration as a function of \( g_{12} \) up to the miscibility threshold.

Upon computing the solution, Bogolyubov-de Gennes stability analysis is implemented that perturbs the solutions according to:

\[
\begin{align*}
\psi_1 &= e^{-i\mu_1 t} \left( \psi_{1(0)}(x) + \delta(a(x)e^{i\omega t} + b^*(x)e^{-i\omega^* t}) \right) \\
\psi_2 &= e^{-i\mu_2 t} \left( \psi_{2(0)}(x) + \delta(c(x)e^{i\omega t} + d^*(x)e^{-i\omega^* t}) \right)
\end{align*}
\]  

(5.5)  

(5.6)

Here \( \omega \) represents the linearization eigenfrequency and the vector \((a, b, c, d)^T\) is the linearized eigenvector pertaining to the respective eigenfrequency. The chemical potentials are denoted by \((\mu_1, \mu_2)\), while \( \delta \) is a formal small parameter of the linearization ansatz. \( \Omega \) represents the strength of the trapping potential in the longitudinal vs. the transverse directions (i.e., the ratio thereof) and thus needs to be \( \Omega \ll 1 \) for the reductions to the one-dimensional effective model to be meaningful. In the following, we will assume \( \Omega = 0 \) unless noted otherwise.

Numerical computations are performed by our collaborators using FreeFem++ [167]. The numerical system developed for computing stationary solutions of the Gross-Pitaevskii equation [168] was extended for the two-component system (5.3)-(5.4). We use quadratic \((P^2)\) finite elements with mesh adaptivity, offering high-resolution of the steep gradients in the solution. The Bogolyubov-de Gennes linear eigenvalue problem corresponding to the \( P^2 \) finite element discretization is solved using the ARPACK library.

5.2.5 1d: Dark-Antidark Solitary Waves

We start by considering the scenario of dark-antidark solitons in one spatial dimension. Solutions of this type are represented in Fig. 86 (a) for the in-trap case \((\Omega = 0.025)\) and Fig. 86 (b) the homogeneous case \((\Omega = 0)\). The latter case (without the trap) is shown for reference to classical dark-bright solitons. In the latter, contrary to what is the case here (where it is finite), the background of this second component is vanishing. In what follows, we focus on the case with
trapping potential.

For $g_{12} = 0$ the solution has the form of a regular dark solitary wave coupled to a ground state in the second component. For a finite value of $g_{12}$ the stationary solution develops a bump at the location of the dark soliton dip. Due to the inter-component repulsion, the density dip in the first component leads to an attracting potential well for the second component. Therefore an antidark peak is formed that becomes stronger as $g_{12}$ is increased, while the first component tends to vanish as the miscibility-immiscibility threshold of $g_{12} = 1$ is approached. This trend is clearly seen in Fig. 86 (a). For this case, we have used $\Omega = 0.025$, although similar results have been found for other values of the trap strength.
Figure 86: 1D Dark-Antidark solitary waves. Three examples of the two components in the dark-antidark state for progressively increasing $g_{12}$: (a) in-trap case with trap frequency $\Omega = 0.025 \ll 1$, (b) homogeneous case with $\Omega = 0$. (c) Dependence of the lowest eigenfrequencies scaled by the trap frequency $\Omega$ for the in-trap case. The plot of the real part of eigenvalues: black dots indicate the theoretical prediction for the anomalous mode (see discussion in the text). The colors are there only to visually aid the eye to identify the continuation of the different modes. Plot of the imaginary part: evidence of very low growth of oscillatory instability for $0.71 < g_{12} < 0.87$. 
Remarkably, in the case of the dark-antidark solitary wave family we find the relevant solution to be generally stable (as shown in Fig. 86 (c)) through a wide interval of variation of the $g_{12}$ parameter; an extremely weak oscillatory instability arises for $0.71 < g_{12} < 0.87$, that will be discussed further below, yet its growth rate is so small that we do not expect it to affect the dynamics in an observable way over the time scales of interest. In the large chemical potential limit, we, in fact, have a detailed handle on the spectrum of the relevant eigenfrequencies in an analytical form. When $g_{12} = 0$, the second component is uncoupled from the first and its spectrum in the ground state consists of eigenfrequencies

$$\omega = \sqrt{\frac{n(n+1)}{2}} \Omega$$

\hspace{1cm} (5.7)

where $n$ is a non-negative integer, as discussed in [169,170]. The dark soliton (DS) spectrum on the other hand, as explained, e.g., in [138] consists of the spectrum of the ground state in which the soliton is “embedded”, given by Eq. (5.7), as well as an extensively studied anomalous (or negative energy) mode associated structurally with the excited state nature of the DS state, and practically with its oscillation inside the parabolic trap. As is well known, the latter mode has the frequency $\omega = \Omega/\sqrt{2}$ in the large chemical potential limit [165,166] and is the lowest excitation frequency in the system. Using an argument similar to that presented in [52], we expect that the relevant mode scales as $\omega \approx \Omega \sqrt{\delta g/(2g)}$, where $\delta g = g - g_{12}$. This prediction is also represented in Fig. 86 (c) (black dots in the plot of the real part of eigenvalues) and is in reasonable agreement with the corresponding numerical result throughout the interval of variation of $g_{12}$. As $\Omega \rightarrow 0$ this mode tends to 0, restoring the translational invariance in the limit when the trap is absent. In the presence of the trap, there are two pairs of modes at $\omega = 0$ associated with the invariance of both components with respect to phase and physically associated with the conservation of atom numbers in both of the nonlinearly coupled components. Regarding the rest of the spectrum (the modes associated with the ground state in each component), it can be seen that they can be partitioned into two fundamental categories, namely those that are essentially left invariant and those that undergo a rapid monotonic decrease (which is nearly linear for small $g_{12}$) as $g_{12}$ increases. The
former ones of these modes are referred as density modes in [137], while the latter ones (decreasing proportionally to \( \sqrt{(g - g_{12})/(g + g_{12})} \)) are referred to as spin modes. As this takes place, it is in principle possible for the anomalous mode (of dark soliton oscillation) and the lowest frequency associated with the ground state to collide and lead to a resonant eigenfrequency quartet [138]. This does happen in the example of Fig. 86 (c) (plot of the imaginary part of eigenvalues) for 0.71 < g_{12} < 0.87, yet as mentioned above the growth rate of this instability is minuscule, and hence it will not be considered further herein.

\[ \text{5.3 THREE-COMPONENT SOLITONS} \]

\[ \text{5.3.1 Introduction} \]

In the previous section, we studied the Dark-Anti-Dark solitons and provided the numerical results that confirm the existence of this new class of solitons in dilute gas BECs. Dark and Dark-Bright (DB) solitons have been extensively studied in the past by several groups including ours [51, 148–151]. Due to their exotic nature and relatively limited number of experimental studies, solitons are naturally of interest to both experimentalists and theorists.

In this section, we provide experimental evidence for the existence of two novel types of solitons. As observed before, the DB solitons, normally consist of two atomic states. For example two \( F = 1 \) Zeeman sublevel states or two Zeeman sublevels of \( F = 1 \) and \( F = 2 \) hyperfine states of \( ^{87}\text{Rb} \) have been widely used. In the following, we use all three Zeeman sublevels of \( F = 1 \) states to form a three-component soliton. In the following experiments, we explicitly look for three-component structures such as Dark-Bright-Bright (DBB) and Dark-Dark-Bright (DDB) components and their stability.

\[ \text{5.3.2 Experimental procedures and considerations} \]

We start with a condensate with approximately \( 10^6 \) \(^{87}\text{Rb} \) atoms in the dipole trap with trapping frequencies \((\omega_x, \omega_y, \omega_z) = 2\pi(1.4, 172, 170) \) Hz. A 45 G magnetic bias field is used for separating the Zeeman levels by approximately 31.5 MHz. The two RF transitions \(|1, -1\rangle \leftrightarrow |1, 0\rangle \) and \(|1, 0\rangle \leftrightarrow |1, 0\rangle \)
$|1, 1\rangle$ differ by 291 kHz due to the quadratic Zeeman shift. Similar to the procedures performed in the previous section, we need to make sure that the magnetic field gradient is minimized across the elongated BEC. That is done by applying the magnetic field for more than 15 s and performing a ToF Stern-Gerlach imaging procedure to vertically separate the atomic species. Any relative horizontal displacement between the atomic species can indicate a magnetic field gradient and should be canceled by adjusting the current in one of the “little bias coils”. The bias magnetic field is generated by combining the fields from the little bias coils and the magnetic trap bias coils. This is mostly due to the fine adjustment capability of the little bias coils. It is worth noting that the active magnetic field stabilization mechanism can not be used in these sequences since the bias magnetic field is changed during the experiment.

Depending on the type of the desired soliton, a different procedure is used for generating the soliton since different soliton species require different population distributions across the Zeeman levels. These populations can be controlled by modifying the Radio Frequency (RF) Adiabatic Rapid Passage (ARP) applied to the cloud that generates the mixture starting from a BEC in the $|1, -1\rangle$ state. For example, longer, more powerful, and narrower ARPs transfer more atoms across the states. Although the transitions are approximately 291 kHz apart, we observe that even relatively weak RF ARPs can populate a wrong state. To avoid this, we use weaker and narrower ARPs, up to the point where no undesired atomic state occupation is observed.

**Dark-Bright-Bright (DBB) Solitons**

To generate the DBB solitons, we start with an elongated condensate in the dipole beam. As mentioned above, a 45 G bias magnetic field is applied to the condensate to break the degeneracy of the Zeeman levels. It is observed that a large magnetic field may prevent spin-exchange collisions due to the larger quadratic Zeeman shift. The magnetic field is generated by combining the bias coils and the little bias coils. The little bias coils are ramped on during the dipole evaporative cooling sequence to minimize excitations in the condensate. The bias coils are turned on after the evaporative cooling sequence and a 6 s wait time is applied for the excitations to dissipate. This is
also very important to make sure that the current in the coils stabilizes. In our default setup, the magnetic fields of the little bias coils and the bias coils happen to oppose each other in the normal configuration. To avoid crossing the zero magnetic field and unwanted spin-exchange collisions, the polarity of the little bias coils is manually reversed.

Next, a weak (−15 dBm), narrow (31.9626 MHz → 32.1626 MHz), and relatively slow (10 ms) RF ARP is applied to the cloud originally in the $|F = 1, m_F = -1\rangle$ state to transfer approximately 10% of the atoms to the $|F = 1, m_F = 0\rangle$ state. Then, a 50 – 100 mG/m magnetic field gradient is applied for approximately 2 s until DB solitons are formed. Up to this point, the procedure is similar to the DB soliton generation procedure. After the DB solitons are formed, we perform another weak ARP (−10 dBm, 31.866 MHz → 31.666 MHz in 10 ms) to transfer 50% of the bright component of the DB soliton to the $|F = 1, m_F = 1\rangle$ state. After the transfer is performed, we wait for a short time to verify the stability of the solitons. We observe a few hundred milliseconds of lifetime for these structures. An example for an observed DBB soliton is presented in Fig. 87.

We start with presenting a ToF experimental image of DBB soliton shown in Fig. 87. It shows a dark notch in the $|F = 1, m_F = -1\rangle$ state and two bright components in the $|F = 1, m_F = 0\rangle$ and $|F = 1, m_F = +1\rangle$ states. The result shown in Fig. 87 has a 100 ms wait time after the soliton is generated to show that the structure is stable. 20 out of 27 trials showed stable DBB solitons structures in them.

![Figure 87: Logarithmic scale of a Stern-Gerlach ToF image of a DBB soliton. There is a 100 ms wait time after the DBB soliton is generated to show the stability of the structure.](image)
Dark-Dark-Bright (DDB) Solitons

To generate the DDB solitons, after turning on the bias magnetic field similar to the procedure for the DBB solitons, a relatively strong ARP is applied (3 dBm, 31.9626 MHz $\rightarrow$ 32.1626 MHz in 10 ms) to transfer all the atoms to the $|F = 1, m_F = 0\rangle$ state. Next, a weak ARP ($-14$ dBm, 31.866 MHz $\rightarrow$ 31.666 MHz in 10 ms) is applied to transfer approximately 5% of atoms to the $|F = 1, m_F = 1\rangle$ state. Next, a small magnetic field gradient is applied and after 2 s wait time, the DB solitons are formed. The dark and the bright components appear in the $|F = 1, m_F = 0\rangle$ and $|F = 1, m_F = 1\rangle$ atomic states respectively. Once the DB solitons are formed, a relatively strong ARP ($-8$ dBm, 31.9626 MHz $\rightarrow$ 32.1626 MHz in 10 ms) is employed to transfer approximately 50% of the atoms in the $|F = 1, m_F = 0\rangle$ to the $|F = 1, m_F = -1\rangle$ state.

Figure 88 shows an image of the Dark-Dark-Bright soliton using the Stern-Gerlach ToF imaging sequence similar to Fig. 87. A 100 ms wait time is applied after the DDB soliton is formed to verify its stability. We observe a relatively high fidelity in these experiments as well. In total of 20 experiments, 14 stable DDB solitons were formed.

![Logarithmic scaled ToF image of DDB solitons after 100 ms wait time. The image is taken using the Stern-Gerlach imaging procedure.](image)

These structures seem to form for a variety of populations of the atomic states. For example, Fig. 89 shows two extreme situations where the population of the $|F = 1, m_F = -1\rangle$ is significantly modified, but the structures are still stable.
Figure 89: Versatility and robustness of DDB solitons is demonstrated. A large variation of the relative atomic population is allowed between $|1, -1\rangle$ and $|1, 0\rangle$ atomic states. (a) has 31%, 63%, and 6% population for $|1, -1\rangle$, $|1, 0\rangle$, and $|1, +1\rangle$ states, respectively. (b) has 71%, 25%, and 4% population for $|1, -1\rangle$, $|1, 0\rangle$, and $|1, +1\rangle$ states, respectively.

5.4 CONCLUSION

In this chapter, we provided experimental evidence for the existence of novel solitonic states in dilute Bose gases. First, we presented the experimental results regarding dark-anti-dark solitons which are very similar in nature to the magnetic solitons proposed by Qu et. al (2016). We measured the oscillation frequency of these solitons in a harmonic trap and observed significantly slower frequencies compared to the trap frequency. A theoretical model was presented confirming the stability of the dark-antidark solitons. Next, we presented experimental evidence for the existence of three-component solitons using the three Zeeman sublevels of the ground state $F = 1$ hyperfine manifold. These results open the door for more in-depth study of these novel solitons. For example, one can study the interactions of the three-component solitons or study their oscillation period in
trap. In particular, it is interesting to study the effect of the number of atoms in each state on the trapping frequency of the three-component solitons.
CHAPTER 6

SUMMARY AND OUTLOOK

In this dissertation, we start with the spin-orbit coupling (SOC) of Bose-Einstein condensates using Raman beams and studying their excitation spectrum. It is shown that a Roton-like mode exists in the spectrum and can be adjusted by tuning the Raman laser parameters. An important question to ask is whether it is possible to lower the Roton mode energy all the way to the ground state energy. We find that this is not the case for our experimental parameters. This shows that observing the stripe phase in our experimental conditions is not possible or very difficult. However, the methods used in our experiments and the results obtained from them can be employed in our future efforts in the hunt for the stripe phase. For example, being in a more miscible regime is a clear motivation for using Feshbach resonances in these experiments. Also, novel SOC schemes in combination with Feshbach resonances can be proposed facilitating the direct observation of the stripe phase.

The fascinating features of the SOC dispersion relation that leads to the existence of a roton-like feature in the excitation spectrum can also lead to states with negative effective mass. The onset of the negative effective mass in the SOC dispersion causes the expansion to significantly slow down although a mean field pressure is constantly supporting the expansion. By adjusting the dispersion, we can control the asymptotic flow velocity of the superfluid in the negative mass region. These experiments provide valuable information about SOC systems and shed light on previous controversies. BdG analysis of the system shows that the occupation of the states with negative effective mass causes the condensate to become modulationally unstable. Our experimental results are in good agreement with both 1D and 3D GP simulations. The exact dynamics occurring in the negative effective mass region is still under investigation. We conclude that the slowing happens inside the negative effective mass region, and the final momentum depends on the initial conditions of the condensate. What is clear in our results is that the final momentum is beyond the inflection point of the dispersion in contrast with previous findings in optical lattices. More theoretical and
experimental investigation is needed for a better understanding of the nonlinear hydrodynamics inside the negative effective mass region.

We continue studying the SOC BECs by presenting experimental results for combining the SOC with a weak lattice. The lattice geometry is chosen such that its momentum matches the quasimomenta of the two minima in the SOC dispersion to induce coherent mixing between them. This could potentially lead to the realization of the stripe phase. Our final presented work that involves Raman dressed SOC, provides the methods for Raman coupling between the $|1, 1\rangle$ and $|2, -1\rangle$ atomic states. Utilizing the Feshbach resonance between these states together with the Raman dressing might be the key to a magnetic field insensitive SOC scheme with tunable interspecies scattering length.

Furthermore, in the context of SOC, we show that SOC can be realized without the use of the Zeeman states as the pseudo-spins. Instead, the $s$ and $p$ Bloch bands of a static lattice are treated as the pseudo-spins and are coupled via a relatively weak moving lattice. The inverted nature of the $p$- band coupled with the $s$- band provides a double well dispersion which is reminiscent of the Raman dressed SOC dispersion. This means that similar to the dispersion in the Raman dressed SOC, the quasimomentum of the ground states is not at zero. The experiments clearly prove this, and they are in reasonable agreement with the devised theoretical model and the GP simulations. A very interesting route to take for future experiments would be the realization of a 2D lattice scheme. However, the condensate lifetime would be a limiting factor for this configuration. The same scheme can be applied to a degenerate Fermi gas which may lead to the observation of exotic phenomena such as the FFLO phases and the Majorana fermions.

The last chapter of this dissertation is dedicated to presenting experimental evidence for the existence of novel solitonic states such as dark-anti-dark (DAD) solitons, dark-bright-bright (DBB) solitons, and dark-dark-bright (DDB) solitons. DAD solitons share similarities with the recently proposed magnetic solitons by Qu et al. (2016). The DAD solitons in a harmonic trap show a significantly slower oscillation compared to the dark-bright solitons. The oscillation period of the DAD solitons does not exactly match that of the magnetic solitons, but this is understood given
the fact that the DAD solitons have a slightly different spin composition. Finally, we present experimental proof for the existence of three species solitons (DBB and DDB) and we show that these structures are stable. Observation of novel solitons motivates both experimentalists as well as theorists in their quest for observing, proposing, and identifying more solitons contributing to this vibrant field of research. Studying the scattering and interactions between these novel multicomponent solitons, both theoretically and experimentally, is a great motivation for our future investigations.
Appendix A

GENERAL PURPOSE TEMPERATURE CONTROLLER UNIT

Figure 90: The temperature controller schematic and the Printed Circuit Board are shown in the figure. The schematic circuit design is by G. Forrest Cook, and I designed the PCB using Eagle PCB design software.
Appendix B

OPTIMIZED MATLAB 3D FUNCTION FOR FINDING THE
DIFFRACTION INTENSITY AT AN ARBITRARY POSITION

%% eulerangle (euangle)
% This function receives the Euler angles as parameter and gives a
% transformation matrix that transforms the coordinates from the unprimed
% coordinate system to the primed coordinate system.
%%
function out = eulerangle (euangle)
alpha = euangle (1);
beta = euangle (2);
gamma = euangle (3);

out = [
    cos (gamma) * cos (beta) * cos (alpha) - sin (gamma) * sin (alpha) * cos (gamma) * cos (beta) * sin (alpha) + sin (gamma) * cos (alpha) - cos (gamma) * sin (beta) ...
    ; -sin (gamma) * cos (beta) * cos (alpha) - cos (gamma) * sin (alpha) - sin (gamma) * cos (beta) * sin (alpha) + cos (gamma) * cos (alpha) * sin (gamma) * sin (beta) ...
    ; sin (beta) * cos (alpha) * sin (beta) * sin (alpha) * cos (beta)];
end

%% Function beam intensity (x,y,z)
% This function calculates the beam intensity at an observation point (x,y,z)

% E_inc: source electric field: Matrix
% lambda: laser wavelength: Scalar
% F: lens focal length: Scalar
% euler_angle: Euler’s angles connecting primed and unprimed coordinates:
% Vector
% r: observation point in the unprimed coordinate system: Vector
% xx_p, yy_p: 2D matrices of coordinates - copied matrices of x and y:
% Matrix
% dx_p, dx_y: differential elements separating small spherical oscillators
% on the source
%%% 
function out = Intensity(E_inc, lambda, F, euler_angle, r, xx_p, 
    yy_p,dx_p, dy_p)
    k = 2*pi/lambda;
    eta = 376.730;

    r_p = eulerangle(euler_angle)*r; % finding the observation point
    % in the
    % primed coordinate system. The beam coordinate system nests the
    % beam
    % propagating in the z.
    % to avoid division by zero:
    if (abs(r_p(3))<1E-8)
        r_p(3) = 1E-8;
    end
    r_p(3) = F-r_p(3); % dsplacing the source the focul point to the
    % origin
    % and the lens by the focul length F from the unprimed
    % coordinate system
    % origin.
    r_p_length = sqrt(r_p'*r_p); % distance from the lens to
    % observation point
    q = r_p(3);
    r_pq = r_p*(F/(F-q)); % distance of the image of the screen to
    % lens

    r_pq_pp_x = r_pq(1)-xx_p;
    r_pq_pp_y = r_pq(2)-yy_p;
    r_pq_pp_z = r_pq(3);

    % delta_d is the path difference that is picked up due to the
    % angle
    % theta. It is is found by finding inner product of the r_pp
    % with the
    % line connecting point g to (x^\prime,y^\prime).
    delta_d = ((xx_p.*r_pq_pp_x)+(yy_p.*r_pq_pp_y))./...
        sqrt(r_pq_pp_x.^2+r_pq_pp_y.^2+r_pq_pp_z.^2);
    Green_fun = exp(1i*k*delta_d)/r_p_length;
    out = abs(sum(sum(Green_fun.*E_inc))*dx_p*dy_p)^2/(2*eta*lambda^2);
end
Appendix C

SATURATION ABSORPTION SPECTROSCOPY

For the MOT to work, a narrow band laser (≈ 100 kHz) is used to keep the atoms in the MOT cycle. To lock the lasers, we use saturation spectroscopy which can resolve atomic transitions well below the Doppler limit. Let us see step by step how saturation spectroscopy works. When an atomic gas is in equilibrium at temperature $T$, the atoms have Maxwell-Boltzmann velocity distribution which is given by

$$f(\vec{v}) = (2\pi mk_BT)^{-3/2} \exp\left\{-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2k_BT}\right\}, \quad (C.1)$$

where $T$ is the temperature, $m$ is the atomic mass, and $\vec{v}$ is the atom velocity. After integrating $v_y$ and $v_z$ out, the velocity distribution becomes

$$f(v_x) = \left(\frac{m}{2\pi k_BT}\right)^{1/2} \exp\left\{-\frac{mv_x^2}{2k_BT}\right\}, \quad (C.2)$$

which is a Gaussian distribution centered around $v_x = 0$.

To study the saturation spectroscopy more intuitively, we go to the atom’s moving frame. The laser frequency is constantly scanned, $\omega(t)$, as measured in the lab frame to perform the spectroscopy. The saturation absorption spectroscopy setup is used e.g. in Fig. 16. The Rb absorption cell has $^{87}\text{Rb}$ atoms in it that interact with the laser beams. Since there are two counter-propagating lasers shining onto the atoms, in the atom frame, every atom observes two laser frequencies $\omega(t) \pm kv_x$ in which $k$ is the laser wave vector. Assume that the atom has two excited states: state 0 and state 1 (see Fig. 91). Therefore, there are two possible atomic transitions for an atom in the ground state $|g\rangle$. 
The laser beam frequencies, $\omega(t) \pm kv_x$, are visualized in Fig. 92 (a) with the red and blue shifted photons represented by the red and blue lines. Two arbitrary atomic transitions are shown as two Lorentzian curves in black at frequencies $\omega_0$ (state 0) and $\omega_1$ (state 1). Without loss of generality, let’s assume that the red-shifted photon is the pump light (with larger intensity), and the blue-shifted photon is the probe. In saturation spectroscopy, we observe the intensity of the probe beam after it passes through the sample. In Fig. 92 (a), when the pump beam goes into resonance with $\omega_0/\omega_1$, the probe beam is not in resonance with either of them and therefore the probe beam intensity remains unchanged. However, the situation is different for the atoms with $v_x \approx 0$ as shown in Fig. 92 (b). In this case, both the pump (red) and the probe (blue) photons are in resonance with the transitions. Since the pump beam is the strong one, it excites a significant number of atoms from the ground state to the excited state (burns a hole into the distribution) and therefore, the probe beam is scattered less by the atoms. Another situation that can occur is for the atoms with $k|v_x| = \frac{\omega_1 - \omega_0}{2}$. In this case, the pump beam excites the atoms to the excited state 0 and it deprives the ground state (burns a hole). At the same time, the probe is in resonance with state 1, but since the ground state is deprived to fill the state 0, the probe beam interacts much less with the atoms and its intensity increases. This is illustrated in Fig. 92 (c). So far, we have investigated the situation for single atoms. An ensemble of atoms is slightly different as we have to take the statistics into account. This can be done as follows: We go to the frame of reference of every single atom in the gas, record the frequency of the red and the blue photons. After doing so for all of the atoms, we find the distribution of the photon frequencies. We observe
a Doppler-broadened Gaussian curve as presented in Fig. 92 (d). As the laser frequency, $\omega(t)$, is scanned across the transitions, the only time that we detect a transition is when both the red and the blue photons contribute to the resonance which only happens at $\omega_0$, $\frac{\omega_0 + \omega_1}{2}$, $\omega_1$. $\omega_0$ and $\omega_1$ are actual atomic transitions. However, $\frac{\omega_0 + \omega_1}{2}$ is a crossover transition that occurs when the pump laser excites many atoms to the excited state 0, and therefore the ground state is deprived. Since the ground state is deprived, the probe laser does not significantly interact with the atoms and therefore a peak in the intensity of the probe beam is observed.

Figure 92: Schematic illustration of saturation spectroscopy.


[74] S.J.J.M.F. Kokkelmans. personal communication. For the states |↓⟩ = |1, −1⟩ and |↑⟩ = |1, 0⟩ used in the experiment, the scattering lengths are a_{↓↓} = 100.4a_0, a_{↓↑} = 100.4a_0 and a_{↑↑} = 100.9a_0, where a_0 is the Bohr radius.


