A One-Dimensional Nonlinear Stability Analysis of Vegetative Pattern Formation for an Interaction-Diffusion Plant-Surface Water Model System in an Arid Flat Environment

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Abstract

The development of spontaneous stationary vegetative patterns in an arid flat environment is investigated by means of a one-dimensional weakly nonlinear diffusive instability analysis applied to the appropriate model system for this phenomenon. In particular, that process can be modeled by a partial differential interaction-diffusion equation system for the plant biomass density and the surface water content defined on an unbounded flat spatial domain. The main results of this analysis can be represented by closed-form plots in the specific rate of precipitation versus the specific rate of plant density loss parameter space. From these plots regions corresponding to bare ground, striped vegetative patterns, and homogeneous distributions of vegetation, respectively, may be identified in this parameter space. Then those theoretical predictions are compared with both relevant observational evidence involving tiger bush patterns and existing numerical simulations of similar model systems as well as placed in the context of the results from some recent nonlinear vegetative pattern formation studies.

Research Area

The vegetative pattern formation in arid flat environments modeled by an interaction-diffusion model system is investigated by nonlinear stability analyses applied to the model system.

- **Previous Work**: Other studies of the Turing-like patterns that occur in semi-arid and arid environments (Africa, Australia, Mexico, Middle East)
  - Arid ecosystems are most prominent in self-organized patchiness
  - Flat ground yields stationary irregular mosaics (primarily stripes)
  - Patterns based on vegetation variation (grass, shrubs, trees), vegetation density, rainfall, soil, and water infiltration

Interaction-Diffusion Model System

Our system is an improved upon version of a pair of partial differential equations found in Klausmeier (1999).

- Let (X, Y) be defined on an infinite two-dimensional domain. Define W = surface water, N = plant biomass, τ = time.

$$\frac{\partial N}{\partial \tau} = F(W,N) + D_1\nabla_2^2 N$$
$$\frac{\partial W}{\partial \tau} = G(W,N) + D_2\nabla_2^2 W$$

where

$$\nabla_2^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$$

For simplicity in our techniques, we define

$$F(W,N) = RJWN^2 - MN$$
$$G(W,N) = A - LW - RWN^2$$

- Water is supplied uniformly at rate A and is lost due to evaporation at rate LW
- Plants take up water at rate $Rf(W)g(N)N$, where $f(W)$ is the functional response of plants to water and $g(N)$ describes how plants increase water infiltration
  - For simplicity we take $f(W) = W$ and $g(N) = N$ (linear)
- J is the yield of plant biomass per unit water consumed
- MN is the density-independent mortality and maintenance rate through which plant biomass is lost
- D_1 and D_2 are the diffusion coefficients

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Bare Ground Equilibrium Point

We find the equilibrium points of this system by considering $F(W_e, N_e) = 0$ and $G(W_e, N_e) = 0$, which yields two possible stable points:

$$N \equiv 0, \quad W \equiv \frac{A}{L}$$

corresponding to a bare ground or no vegetation situation that always exists and is always stable.

Homogeneous Vegetation Equilibrium Point

$$N \equiv N_e = \frac{AJ}{2M} + \left[\left(\frac{AJ}{2M}\right)^2 - \left(\frac{L}{R}\right)\right]^{1/2}$$
$$W \equiv W_e = \frac{M}{RJN_e}$$

corresponding to a situation of homogeneous vegetation that exists when $\left(\frac{AJ}{2M}\right)^2 \geq \frac{L}{R}$ and the stability of which is the primary focus of this research.

Nondimensionalizing

Let

$$n = \frac{N}{N_e}, \quad w = \frac{W}{W_e}, \quad t = L\tau, \quad (x,y) = \left(\frac{L}{D_2}\right)^{1/2}(X,Y)$$
$$a = \frac{AR^{1/2}J}{L^{3/2}}, \quad \alpha = \frac{M}{L}, \quad \mu = \frac{D_1}{D_2}$$
$$\nu = \frac{a}{2\alpha}, \quad \beta = \frac{R}{L}N_e^2 = (\nu + \sqrt{\nu^2 - 1})^2, \quad \text{where } \nu \geq 1$$

Our new system:

$$\frac{\partial n}{\partial t} = \alpha w n^2 - \alpha n + \mu \nabla^2 n$$
$$\frac{\partial w}{\partial t} = 1 + \beta(1 - wn^2) - w + \nabla^2 w$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Stuart-Watson Analysis

Let

$$n(x,t) = 1 + n_1 A(t) + n_2 A^2(t) + n_3 A^3(t) + O(A^4)$$
$$w(x,t) = 1 + w_1 A(t) + w_2 A^2(t) + w_3 A^3(t) + O(A^4)$$

where

$$\frac{dA}{dt} = \sigma A(t) - a_1 A^3(t) + O(A^5)$$

Let

$$w_1(x) = w_{11}\cos(qx), \quad n_1(x) = n_{11}\cos(qx),$$
$$w_2(x) = w_{20} + w_{22}\cos(2qx), n_2(x) = n_{20} + n_{22}\cos(2qx)$$
$$w_3(x) = w_{31}\cos(qx) + w_{33}\cos(3qx)$$
$$n_3(x) = n_{31}\cos(qx) + n_{33}\cos(3qx)$$

Making the necessary substitutions, using some trigonometric identities, and collecting terms by order, we can obtain the following...

Linear Stability Analysis

For $(1+\mu)q^2 + \beta + 1 - \alpha > 2 - \alpha > 0$, we obtain the Turing diffusive instability ($\sigma > 0$) by considering

$$\mu q^4 + q^2[\mu(\beta+1) - \alpha] + \alpha(\beta - 1) < 0$$

Then

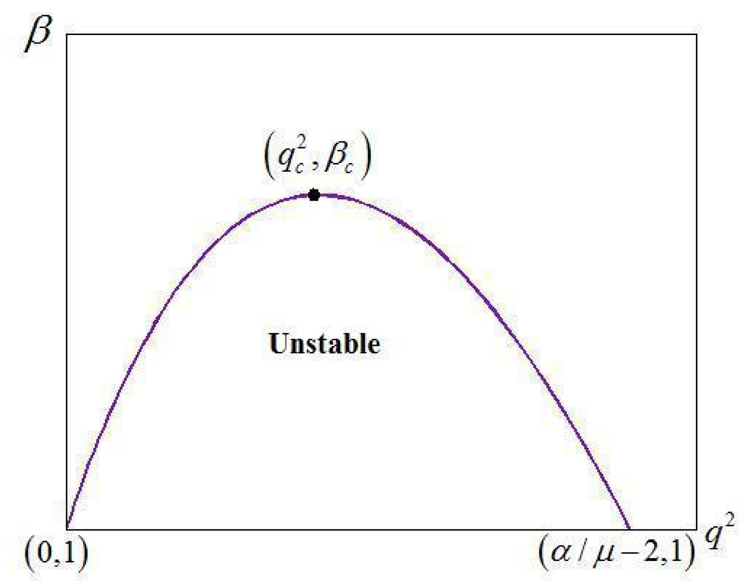
$$\beta < \frac{-\mu q^4 - q^2\mu + q^2\alpha + \alpha}{q^2\mu + \alpha} = \frac{(q^2+1)(\alpha - \mu q^2)}{q^2\mu + \alpha} = \beta_0(q^2)$$

Critical Values:

$$q_c^2 = \frac{-\alpha + \sqrt{2\alpha(\alpha-\mu)}}{\mu}, \quad \beta_c = \frac{(3\alpha - \mu) - 2\sqrt{2\alpha(\alpha - \mu)}}{\mu}$$

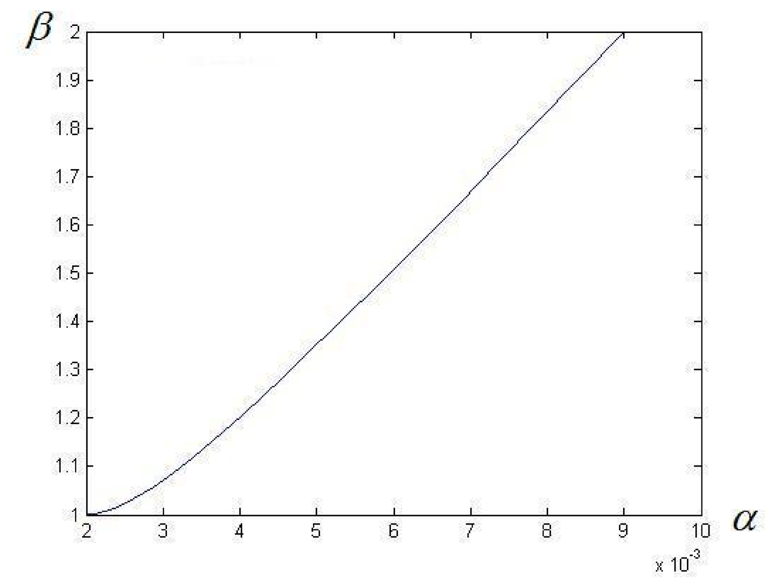
Instability Graph

Note: $\beta > 1$, $\alpha > 2\mu$

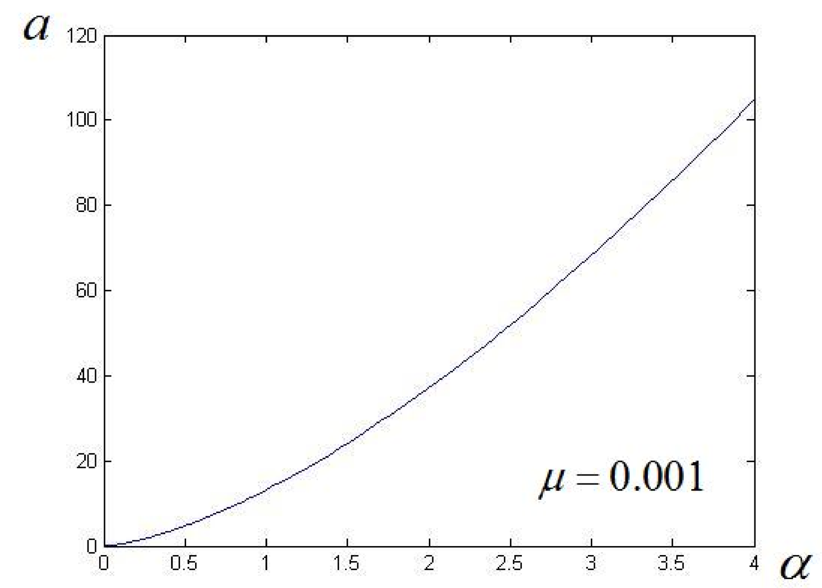


Plot $\beta_c = \dfrac{(3\alpha - \mu) - 2\sqrt{2\alpha(\alpha - \mu)}}{\mu}$

Note: $\beta > 1$, $\alpha > 2\mu$, $\mu = 0.001$

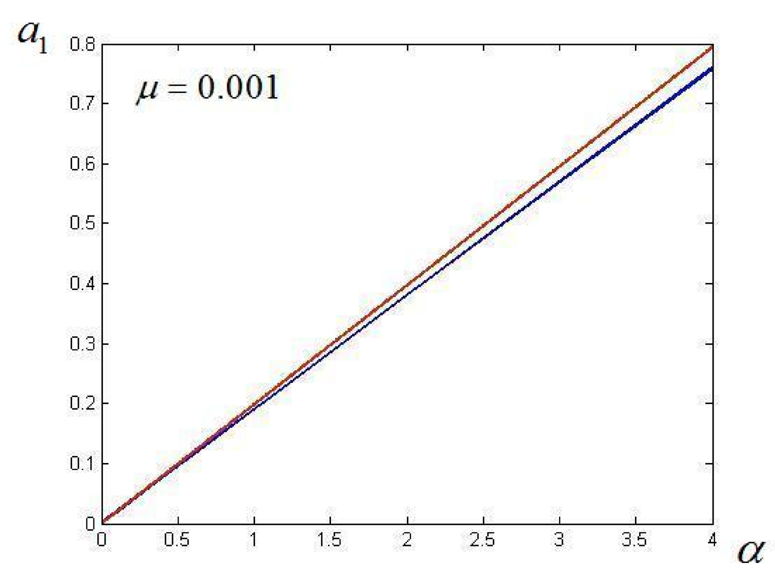


Plot $a = 2\alpha\cosh\left\{\frac{1}{2}\ln\beta_c(\alpha)\right\}$



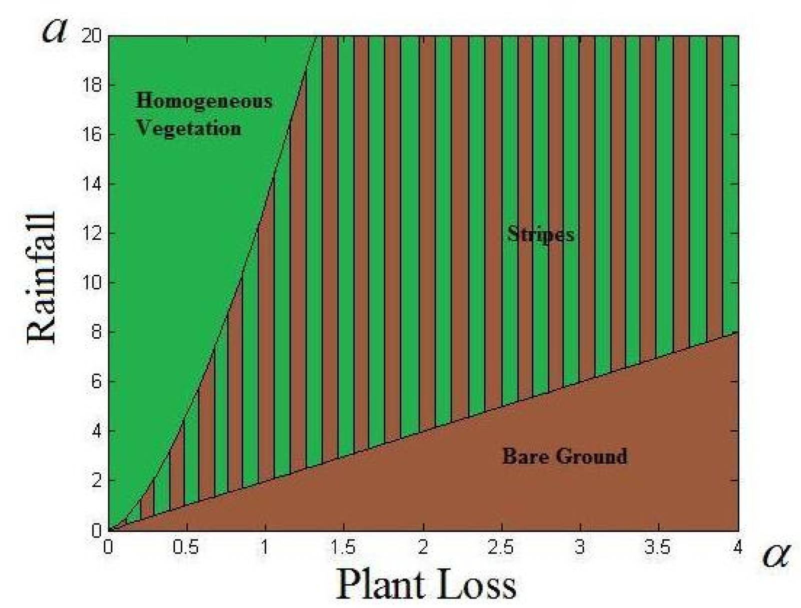
Nonlinear Stability Analysis

Note: $q \equiv q_c$



$$a_1 \sim \frac{10\sqrt{2} - 7}{36}\alpha \quad \text{(red) vs.} \quad a_1 = \left.\frac{-(q_c^2+1)\alpha r_{31}}{\beta + 1 - \alpha + (1+\mu)q_c^2}\right|_{\beta = \beta_c} \quad \text{(blue)}$$
$$r_{31} = 3n_{11}^2 w_{11}/4 + (n_{11} + w_{11})(2n_{20} + n_{22}) + n_{11}(2w_{20} + w_{22})$$

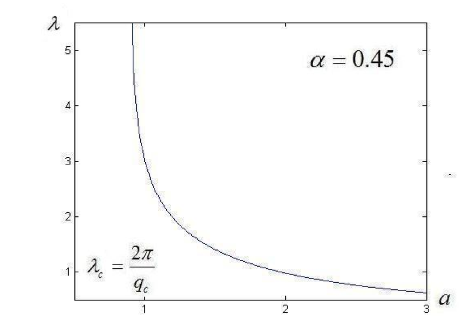
One-Dimensional Pattern Formation Results

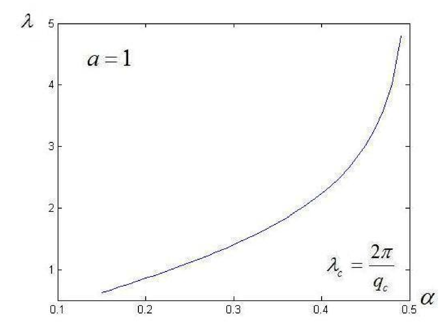


$$a = 2\alpha\cosh\left\{\frac{1}{2}\ln\left[\frac{(3\alpha - \mu) - 2\sqrt{2\alpha(\alpha - \mu)}}{\mu}\right]\right\}$$
$$a = 2\alpha, \quad \mu = 0.001$$

Wavelength

$$\beta = \beta_c(\alpha) \quad \Rightarrow \quad \frac{\alpha}{\mu} = 3\beta - 1 + 2\sqrt{2\beta(\beta-1)} \quad \Rightarrow \quad q_c^2 = \beta - 1 + \sqrt{2\beta(\beta - 1)}$$
$$\lambda_c = \frac{2\pi}{q_c}, \quad \lambda_c^* = \lambda_c\left(\frac{D_2}{L}\right)^{1/2} = 95.5\lambda_c m \text{ for } D_2 = 100m^2/\text{d and } L = 4/\text{yr}$$





Comparison: Lefever and Lejeune (1997)

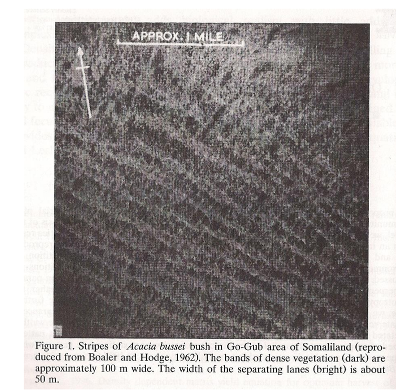


Figure 1. Stripes of *Acacia bussei* bush in Go-Gub area of Somaliland (reproduced from Boaler and Hodge, 1962). The bands of dense vegetation (dark) are approximately 100 m wide. The width of the separating lanes (bright) is about 50 m.

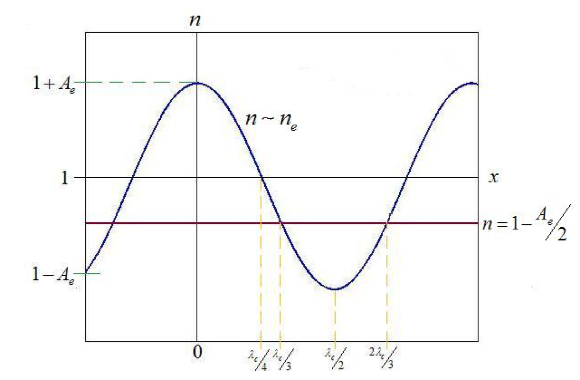
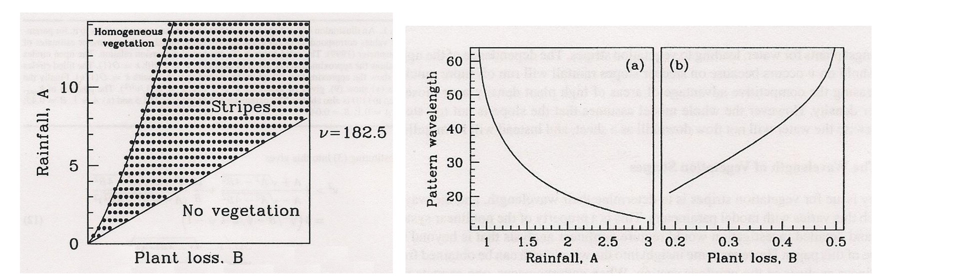


Fig 1: Lefever and Lejeune (1997) $\quad n_e = 1 + A_e\cos\left(\frac{2\pi x}{\lambda_c}\right), \; A_e = \left(\frac{\sigma}{a_1}\right)^{1/2}$

a	α	λ_c	λ_c^*
1	0.325	1.57	150
1.386	0.450	1.57	150

Comparison: Sherratt (2005)

- $D_2\nabla_2^2 W$ replaced by $V\frac{\partial W}{\partial X}$ and $v = \frac{V}{(D_1 L)^{1/2}} \equiv$ advection effect
- Patterns are only predicted for $v > v_c \sim \frac{2\sqrt{2}a^2}{\alpha^{5/2}}$ - i.e., no pattern for flat environments ($v = 0$).



Figures: Sherratt (2005)

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