A Vegetative Pattern Formation Aridity Classification Scheme along a Rainfall Gradient: An Example of Desertification Control

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Research Area

The vegetative pattern formation in arid flat environments modeled by an interaction-diffusion model system is investigated by nonlinear stability analyses applied to the model system.

Interaction-Diffusion Model System

Our system is an extension of a pair of partial differential equations found in Klausmeier (1999).

- Let (X, Y) be defined on an infinite two-dimensional domain. Define W = surface water, N = plant biomass, τ = time.

$$\frac{\partial N}{\partial \tau} = F(W,N) + D_1\nabla_2^2 N$$
$$\frac{\partial W}{\partial \tau} = G(W,N) + D_2\nabla_2^2 W$$

where

$$\nabla_2^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$$

- D_1 and D_2 are the diffusion coefficients for plants and water, respectively

For simplicity in our techniques, we define

$$F(W,N) = RJWN^2 - MN$$
$$G(W,N) = A - LW - RWN^2$$

- Plants take up water at rate $Rf(W)g(N)N$, where $f(W)$ is the functional response of plants to water and $g(N)$ describes how plants increase water infiltration
  - For simplicity we take $f(W) = W$ and $g(N) = N$ (linear)
- J is the yield of plant biomass per unit water consumed
- MN is the density-independent mortality and maintenance rate through which plant biomass is lost
- Water is supplied uniformly at rate A and is lost due to evaporation at rate LW

Bare Ground Equilibrium Point

We find the equilibrium points of this system by considering $F(W_e, N_e) = 0$ and $G(W_e, N_e) = 0$, which yields two possible stable points:

$$N \equiv 0, \quad W \equiv \frac{A}{L}$$

corresponding to a bare ground or no vegetation situation that always exists and is always stable.

Homogeneous Vegetation Equilibrium Point

$$N \equiv N_e = \frac{AJ}{2M} + \left[\left(\frac{AJ}{2M}\right)^2 - \left(\frac{L}{R}\right)\right]^{1/2}$$
$$W \equiv W_e = \frac{M}{RJN_e}$$

corresponding to a situation of homogeneous vegetation that exists when $\left(\frac{AJ}{2M}\right)^2 \geq \frac{L}{R}$ and the stability of which is the primary focus of this research.

Nondimensionalizing

$$n = \frac{N}{N_e},\ w = \frac{W}{W_e},\ t = L\tau,\ (x,y) = \left(\frac{L}{D_2}\right)^{1/2}(X,Y)$$

$$a = \frac{AR^{1/2}J}{L^{3/2}} \text{ (precipitation)},\ \alpha = \frac{M}{L} \text{ (plant loss)}$$

$$\mu = \frac{D_1}{D_2} \text{ (relates diffusion coefficients)}$$

$$\nu = \frac{a}{2\alpha} \text{ (relates precipitation and plant loss)}$$

$$\beta = \frac{R}{L}N_e^2 = (\nu + \sqrt{\nu^2 - 1})^2 \text{ (plant density)}$$

Note: where $\nu \geq 1 \Rightarrow \beta \geq 1 \Rightarrow a \geq 2\alpha$

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Nondimensionalized System

Our new system:

$$\frac{\partial n}{\partial t} = \alpha w n^2 - \alpha n + \mu\nabla^2 n$$
$$\frac{\partial w}{\partial t} = 1 + \beta(1 - wn^2) - w + \nabla^2 w$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Note: The equilibrium point for the nondimensionalized system is $(1,1)$.

Hexagonal Planform Expansion

$$n(x,y,t) - 1 \sim A_1(t)\cos[q_c x + \phi_1(t)] + A_2(t)\cos[q_c(x - \sqrt{3}y)/2 - \phi_2(t)] + A_3(t)\cos[q_c(x + \sqrt{3}y)/2 - \phi_3(t)]$$

where

$$\frac{dA_i}{dt} \sim \sigma A_i - 4a_0 A_j A_k \cos(\phi_i + \phi_j + \phi_k) - A_i[a_1 A_i^2 + 2a_2(A_j^2 + A_k^2)],$$

$$A_i\frac{d\phi_i}{dt} \sim 4a_0 A_j A_k \sin(\phi_i + \phi_j + \phi_k),$$

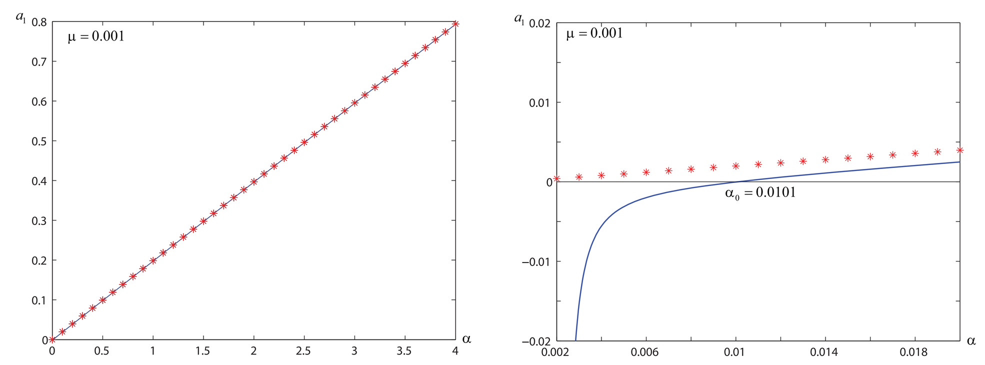
$(i,j,k) =$ even permutation of $(1,2,3)$

with an analogous expansion for $w(x,y,t)$.

One-Dimensional Pattern Formation Results

$$A_2 = A_3 = \phi_1 = \phi_2 = \phi_3 = 0$$

Landau Constant



$$a_1 \sim \frac{10\sqrt{2} - 7}{36}\alpha \text{ (red '*') vs. } a_1 = \frac{-(q_c^2 + 1)\alpha r_{31}}{\beta + 1 - \alpha + (1+\mu)q_c^2}\Bigg|_{\beta = \beta_c} \text{ (blue)}$$

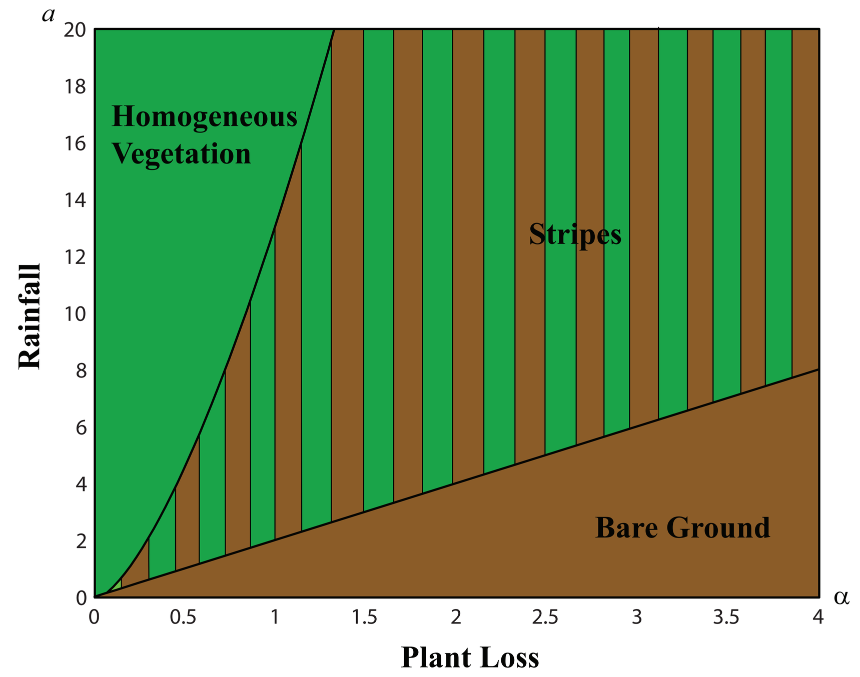
$$r_{31} = 3n_{11}^2 w_{11}/4 + (n_{11} + w_{11})(2n_{20} + n_{22}) + n_{11}(2w_{20} + w_{22})$$

Note: Biologically meaningful values: $\alpha_{\text{tree}} = 0.045$ and $\alpha_{\text{grass}} = 0.45$

- Satisfies constraints: $\alpha > \alpha_0(\mu)$ for $0 < \mu \leq 0.001$
- Hence the zero of a_1 is irrelevant and we will consider a_1 positive.

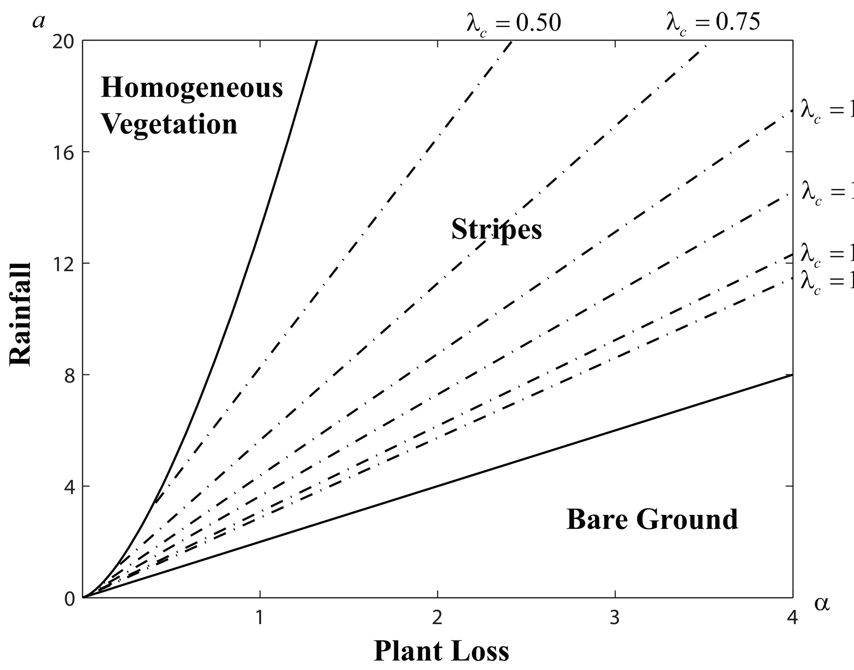
Thus, the amplitude function $A_1(t)$ undergoes a standard supercritical pitchfork bifurcation at $\beta = \beta_c$.

- For $\beta > \beta_c$, the undisturbed state $A_1 = 0$ is stable, yielding a uniform homogeneous vegetative pattern $n(x,t) \sim 1$.
- For $1 < \beta < \beta_c$, $A_1 = A_e = (\sigma_c/a_1)^{1/2} > 0$ is stable, yielding a periodic one-dimensional vegetative pattern consisting of stationary parallel stripes $n(x,t) \sim n_e(x) = 1 + A_e\cos(2\pi x/\lambda_c)$ of characteristic wavelength $\lambda_c = 2\pi/q_c$.



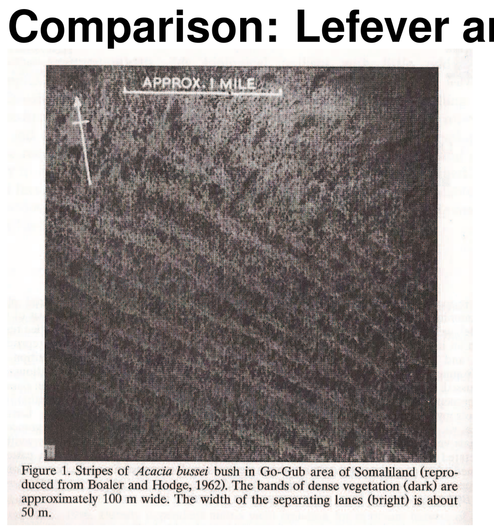
$$a = 2\alpha\cosh\left[\tfrac{1}{2}\ln(\beta_c)\right] = \alpha(\beta_c^{1/2} + \beta_c^{-1/2}),$$
$$a = 2\alpha,\ \mu = 0.001$$

Wavelength

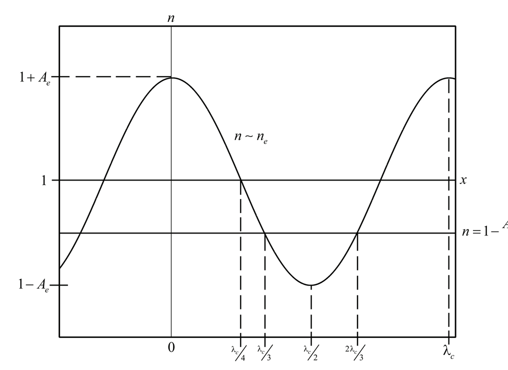


$$\lambda_c^* = \lambda_c\left(\frac{D_2}{L}\right)^{1/2},\ \text{for } D_2 = 100m^2/\text{d},\ L = 4/\text{yr}$$

Comparison: Lefever and Lejune



Lefever and Lejune (1997)

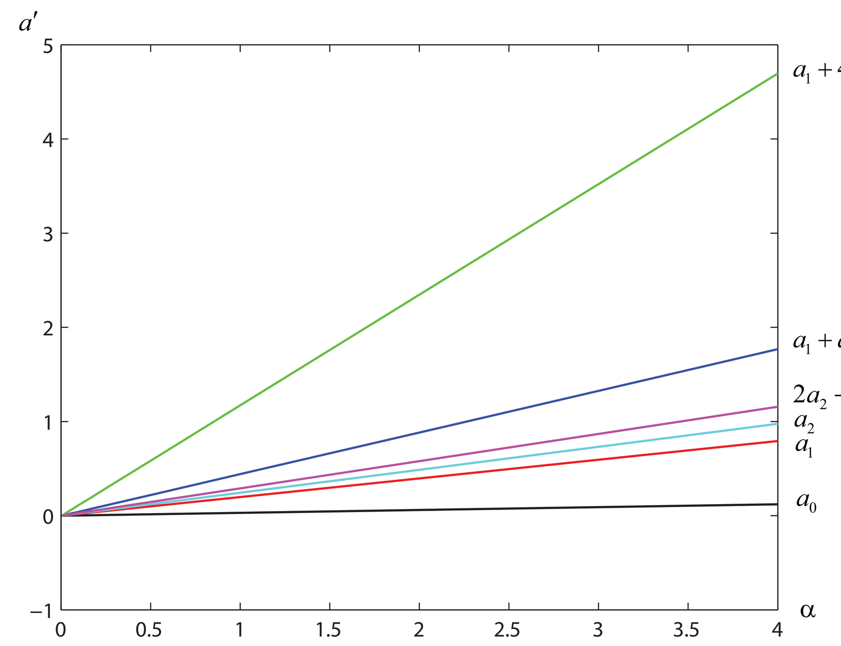


$$n_e = 1 + A_e\cos\left(\frac{2\pi x}{\lambda_c}\right),\ A_e^2 = \frac{\sigma}{a_1}$$

a	α	λ_c	λ_c^*
0.1386	0.0450	1.57	150

Two-Dimensional Pattern Formation Results

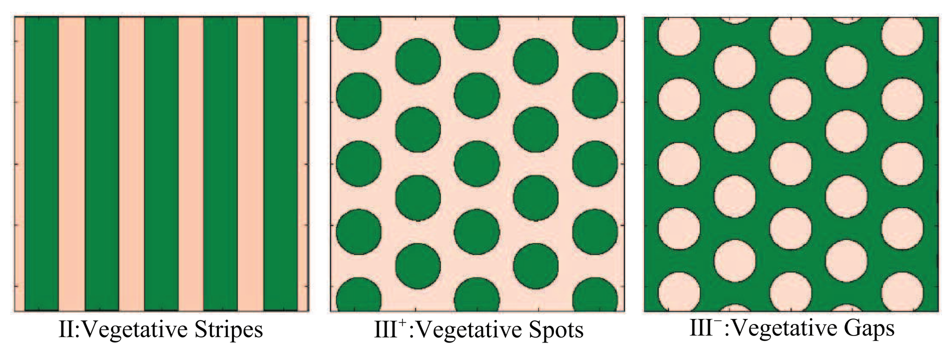
Behavior of the Landau Constants



a_0	a_1	a_2	$a_1 + 4a_2$	$a_1 + a_2$	$2a_2 - a_1$
None	10μ	7μ	7.5μ	8.4μ	4.6μ

Approximate zeroes of the Landau constants

Vegetative Pattern Formations for Hexagonal Planform Analysis

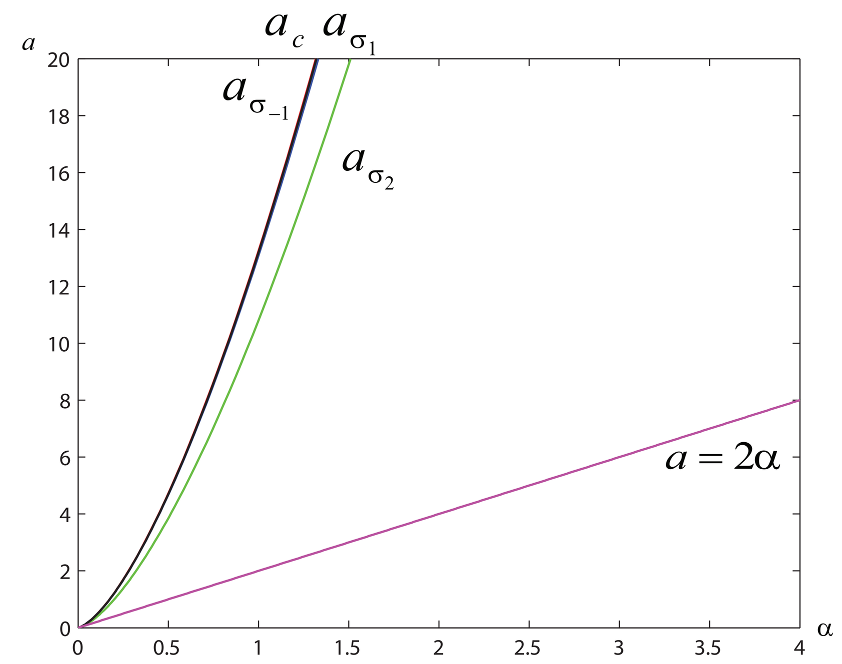


Orbital Stability Behavior

The orbital stability conditions for the critical points can be posed in terms of σ and are summarized in the following table:

a_0	$2a_2 - a_1$	Stable Structures
$+$	$-, 0$	III^- for $\sigma > \sigma_{-1}$
$+$	$+$	III^- for $\sigma_{-1} < \sigma < \sigma_2$, II for $\sigma > \sigma_1$
0	$-$	$\text{III}^{\pm}$ for $\sigma > 0$
0	$+$	II for $\sigma > 0$
$-$	$+$	III^+ for $\sigma_{-1} < \sigma < \sigma_2$, II for $\sigma > \sigma_1$
$-$	$-, 0$	III^+ for $\sigma > \sigma_{-1}$

$$\sigma_{-1} = -4a_0^2/(a_1 + 4a_2),\quad \sigma_1 = 16a_1a_0^2/(2a_2 - a_1)^2,$$
$$\sigma_2 = 32(a_1 + a_2)a_0^2/(2a_2 - a_1)^2$$



$a = \alpha(\beta^{1/2} + \beta^{-1/2})$ where $a_{\sigma_{-1}}, a_{\sigma_1}, a_{\sigma_2}, a_c$, with $a = 2\alpha$

Note: $a_{\sigma_{-1}}$, a_c, and a_{σ_1} are visibly coincident. Hence, for biologically meaningful parameter values, we shall take $a_{\sigma_{-1}} = a_c = a_{\sigma_1}$.

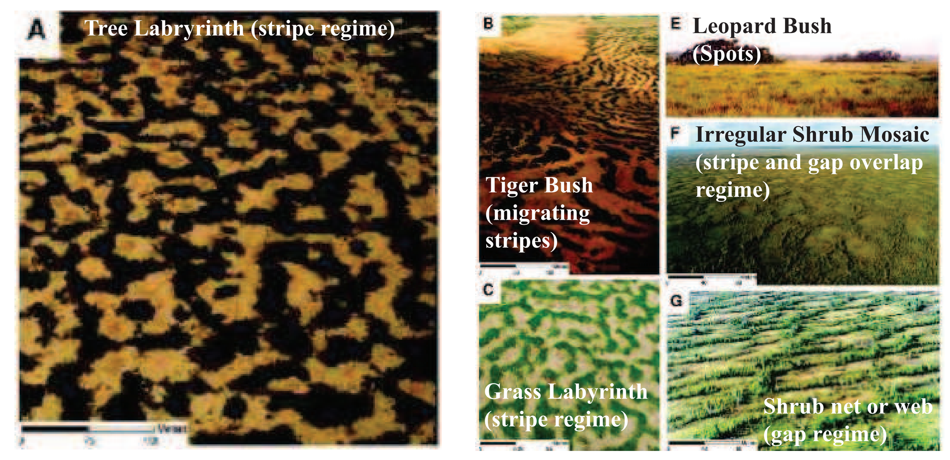
a range	Stable Pattern
$a > a_c$	Homogeneous
$a_{\sigma_2} < a < a_c$	Gaps and Stripes
$2\alpha < a < a_{\sigma_2}$	Stripes
$0 < a < 2\alpha$	Bare Ground

Aridity Classification Scheme

($\alpha = 0.045$ and $\mu = 0.001$)

Dry-subhumid ($a > 0.1442$): Homogeneous
Semiarid ($0.1198 < a < 0.1442$): Gaps and Stripes
Arid ($0.0900 < a < 0.1198$): Stripes
Hyperarid ($0 < a < 0.0900$): Bare Ground

Other Examples of Vegetative Patterns



Bushy Patterns: Niger (A)-(B), Israel (C), French Guiana (E)
Peatlands: Western Siberia (F)-(G)

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- Lefever, R. and O. Lejeune. 1997. On the origin of tiger bush. *Bull. Math. Biol.* **59**, 263-294.